Coexistence in preferential attachment networks
Joint work with Tonći Antunović and Elchanan Mossel

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Alice moves to Berkeley
Alice moves to Berkeley
Model  Two types  Many types

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Main question:

Given this model of product adoption, what will happen to the competing companies?
Coexistence in markets

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Will one company take over the market?  Will the companies coexist?
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**Empirically:** in many markets competing companies *coexist.*

![at&t, verizon, Sprint, T-Mobile](image)
Coexistence in markets

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Main qualitative feature of our model:

In “many” cases companies coexist.
Evolution of graph: **linear preferential attachment model**

- **Initial graph** $G_0 = (V_0, E_0)$.
- **At each time step:**
  - add a new node;
  - add $m$ edges, connecting new node to existing nodes.
- **Edges chosen according to linear preferential attachment**, i.e., for each of the $m$ edges independently let

$$
P(\text{connect to } v) = \frac{\deg(v)}{Z}.
$$
"Extra layer" on top of preferential attachment dynamics.

- Every node in the initial graph has a type / color, out of $N$ possible types / colors.
- When a node is added to the graph, it also gets a type.
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    where $u^i$ is the number of neighbors of type $i$. 

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**Model**

Two types

Many types
Type adoption

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- When a node is added to the graph, it also gets a type.
  - The types of its neighbors can be represented by
    \[
    u = (u^1, \ldots, u^N),
    \]
    where $u^i$ is the number of neighbors of type $i$. Then:
    \[
    P(\text{new node is of type } i \mid u) = p_{i|u}.
    \]
    
- $\{p_{i|u}\}_{u,i}$ are parameters of the model.
Examples

Model Two types Many types

Linear model: \( p^i_u = \frac{u^i}{m} \)

Plurality: \( p^i_u = 1 \left[ i = \operatorname{arg\ max} u^j \right] \)

Don’t listen, pick randomly: \( p^i_u = \frac{1}{N} \)

...or anything else...
Main question:

What are the fractions of nodes of each type?

- Does one type dominate asymptotically?
- Or do types coexist in the limit?
Related work

- **Word-of-mouth recommendations**
  - Strong influence on consumer behavior (Dichter (1966), Goldenberg, Libai, Muller (2001))
  - Online feedback mechanisms (Dellarocas (2003)) and social networks (Leskovec, Adamic, Huberman (2007))

- **Word-of-mouth learning in economics**
  - Ellison, Fudenberg (1995)

- **Competing markets**
Related work

- Epidemiology
  - Under what conditions does the disease die out or take over?
  - Underlying network structure greatly affects the epidemic threshold (Pastor-Satorras, Vespignani (2001))

- Computer science
  - Diffusion of information and opinions
  - Prakash et al. (2012): winner takes all

- Probability theory
  - Competing first passage percolation
  - Antunovic et al. (2011), Deijfen, van der Hofstad (2013): winner takes all
Related work

- Out-of-equilibrium viewpoint of Arthur (80’s—present)
  - Systems with positive feedback due to increasing returns
  - Evolution of technology choice; industry locations

- Commonalities:
  - Qualititatively: multiple possible long-run states, unpredictability, lock-in, path dependence, symmetry breaking.
  - Technically: nonlinear Pólya urn processes.

- Differences:
  - Explicit modeling of underlying network
Focus now on only $N = 2$ types:

- When only two types (red / blue), then parameters are $\{p|_k\}_{0 \leq k \leq m}$:
  \[
  \mathbb{P}(\text{red} | k \text{ red neighbors}) = p|_k
  \]

- $A_n := \# \text{ red nodes}, \quad a_n := \text{fraction of red nodes}$
- $B_n := \# \text{ blue nodes}$
- $X_n := \sum_{v:\text{ red}} \deg(v) = \# \text{ red half-edges}, \quad x_n := \frac{X_n}{X_n + Y_n}$
- $Y_n := \sum_{v:\text{ blue}} \deg(v) = \# \text{ blue half-edges}$
Main results — linear model

Theorem (Linear model)

Assume that \( p_k = k/m \) for all \( 0 \leq k \leq m \), and that \( 0 < a_0 < 1 \).

- \( a_n \) converges a.s.;
- the limiting distribution
  - has full support on \([0, 1]\),
  - has no atoms,
  - depends only on \( X_0, Y_0 \) and \( m \).
Empirical histograms of $a_n$ in the linear model for $n = 10^5$, from $2 \times 10^5$ simulations.
Main results — nonlinear models

Theorem (Nonlinear models)

Assume $p_k \neq k/m$ for some $0 \leq k \leq m$, and $0 < a_0 < 1$. Then

- $a_n$ converges a.s.,
- the limit is a point in the finite set

$$Z_P := \{z \in [0, 1] : P(z) = 0\},$$

where

$$P(z) = \frac{1}{2} \sum_{k=0}^{m} \binom{m}{k} z^k (1 - z)^{m-k} \left(p_k - \frac{k}{m}\right).$$

Why? $a_n$ evolves like a stochastic version of the ODE $dz/dt = P(z)$. 
Proof ideas

\[ \{x_n\}_{n \geq 0} \text{ is a stochastic approximation process with function } P: \]

\[ x_{n+1} - x_n = \frac{1}{n} \left( P(x_n) + \xi_{n+1} + R_n \right), \]

- Introduced in 1951 by Robbins and Monro
- Most results follow from known results about stochastic approximation processes
- In particular:
  - Hill, Lane, and Sudderth (1980)
  - and subsequent refinements by Pemantle
- Variance arguments for \((0, 1)\)
- Domination arguments for the endpoints \(\{0, 1\}\)
Model Two types Many types

Coexistence

\[ m = 3, \ p_{|0} + p_{|3} = 1, \ p_{|1} + p_{|2} = 1 \] (no inherent bias)

Winner can take all iff \( p_{|0} = 0 \) and \( P'(0) < 0 \), or \( p_{|m} = 1 \) and \( P'(1) < 0 \).
Setup

- $N \geq 3$ types / colors
- $A_n = (A^1_n, \ldots, A^N_n)$: number of each type, $a_n$: normalized
- $X_n = (X^1_n, \ldots, X^N_n)$: sum of degrees of each type, $x_n$: normalized
- $\{p^i_u\}_{u,i}$: parameters of the model

Question: asymptotic behavior of $a_n$?
Linear model

Same results as for $N = 2!$

**Theorem (Linear model)**

Assume that $p^i_u = \frac{u^i}{m}$ for all $u, i$, and that $X^i_0 > 0$ for all $i$.

- $a_n$ converges a.s.;
- the limiting distribution
  - has full support on $\Delta^N$,
  - has no atoms,
  - depends only on $X_0$ and $m$. 
Nonlinear models

A key role in the asymptotic behavior of the process \( \{a_n\}_{n \geq 0} \) is played by the vector field

\[
P(z) = \frac{1}{2} \sum_{i=1}^{N} \sum_{u} \binom{m}{u} (z)^u \left[ p^i_u - \frac{u^i}{m} \right] \delta^i,
\]

Conjecture (Nonlinear models)

Assume that \( p^i_u \neq \frac{u^i}{m} \) for some \( u, i \), and that \( X^i_0 > 0 \) for all \( i \).
Then

- \( a_n \) converges a.s.,
- the limit is a point in the set

\[
Z_P := \left\{ z \in \Delta^N : P(z) = 0 \right\}.
\]
Summary and open questions

Takeaways:

▶ Model:
  type adoption coupled w/ preferential attachment dynamics
▶ Can explain coexistence in markets

Open questions:

▶ Limiting density for linear model
▶ $N \geq 3$ types: understanding the vector field $P$
▶ Related models...
Summary and open questions

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- **Model:** type adoption coupled with preferential attachment dynamics
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- Limiting density for linear model
- $N \geq 3$ types: understanding the vector field $P$
- Related models...

Thank you!