Modeling flocks and prices: jumping particles with an attractive interaction

Joint work with Márton Balázs and Bálint Tóth

Miklós Z. Rácz

UC Berkeley

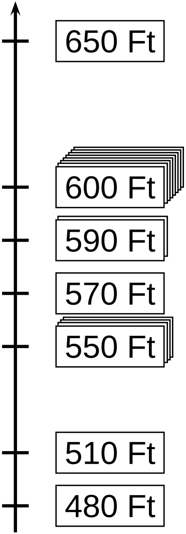
Mathematical Physics and Probability Seminar, UC Davis
14 November, 2012.

Competing prices

- n agents
- Same product

Competing prices

- *n* agents
- Same product
- E.g. *gyros*

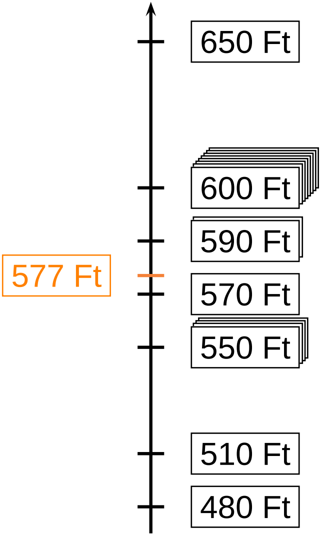


Gyros prices
Budapest, Nov. 2008

Competing prices

Some observations:

- the average price goes up with time

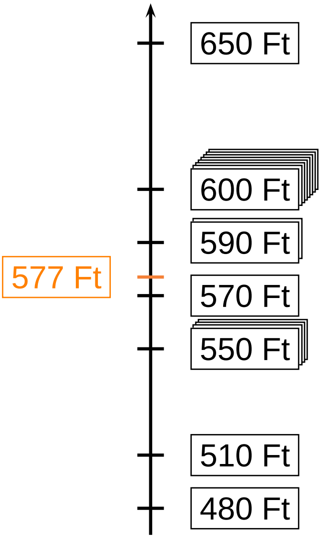


Gyros prices
Budapest, Nov. 2008

Competing prices

Some observations:

- the average price goes up with time
- those who sell at a lower price can raise prices more easily
- those who sell at a higher price cannot raise prices as easily

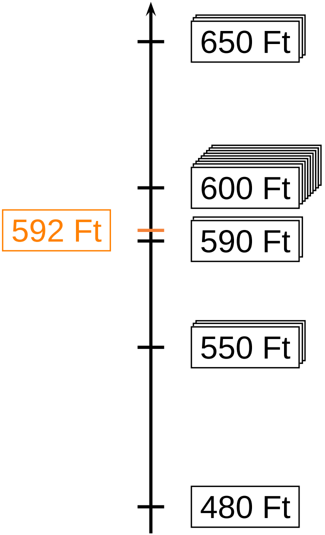


Gyros prices
Budapest, Nov. 2008

Competing prices

Some observations:

- the average price goes up with time
- those who sell at a lower price can raise prices more easily
- those who sell at a higher price cannot raise prices as easily

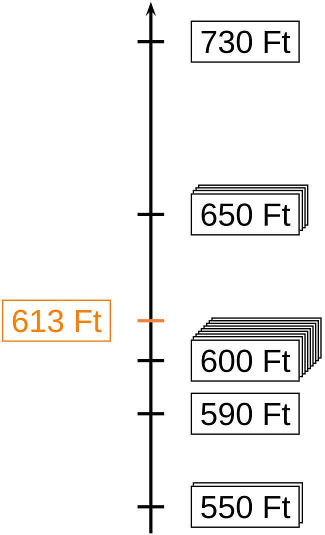


Gyros prices
Budapest, July 2009

Competing prices

Some observations:

- the average price goes up with time
- those who sell at a lower price can raise prices more easily
- those who sell at a higher price cannot raise prices as easily



Gyros prices
Budapest, June 2010

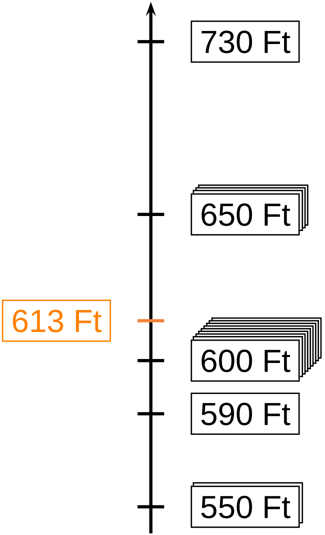
Competing prices

Some observations:

- the average price goes up with time
- those who sell at a lower price can raise prices more easily
- those who sell at a higher price cannot raise prices as easily

Questions:

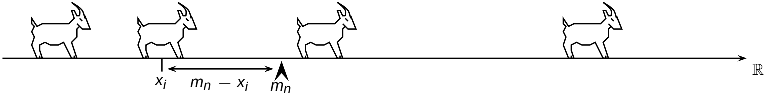
- What is the speed of the rise of the average price?
- What is the distribution of the prices around the average price?



Gyros prices
Budapest, June 2010

The model (Bálint Tóth)

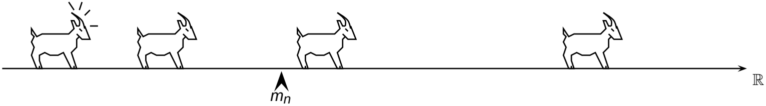
Goats jump on $\mathbb{R}$.



- n goats jump on $\mathbb{R}$ (state space is $\mathbb{R}^n$).
- Given a configuration $x_1, x_2, \ldots, x_n$ of goats the center of mass is $m_n = \frac{1}{n}\sum_{i=1}^{n} x_i$.
- Goat i jumps with rate $w(x_i - m_n)$, where w is the jump rate function: $\mathbb{R} \to \mathbb{R}^+$, decreasing.
- Jump lengths are positive, random, independent of everything, have mean one (and are of density φ).

The model (Bálint Tóth)

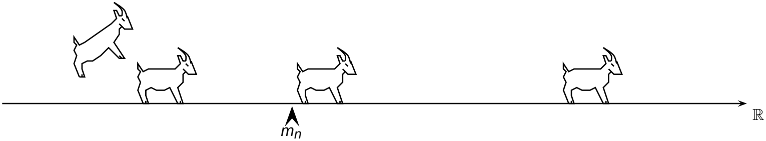
Goats jump on $\mathbb{R}$.



- n goats jump on $\mathbb{R}$ (state space is $\mathbb{R}^n$).
- Given a configuration $x_1, x_2, \ldots, x_n$ of goats the center of mass is $m_n = \frac{1}{n}\sum_{i=1}^{n} x_i$.
- Goat i jumps with rate $w(x_i - m_n)$, where w is the jump rate function: $\mathbb{R} \to \mathbb{R}^+$, decreasing.
- Jump lengths are positive, random, independent of everything, have mean one (and are of density φ).

The model (Bálint Tóth)

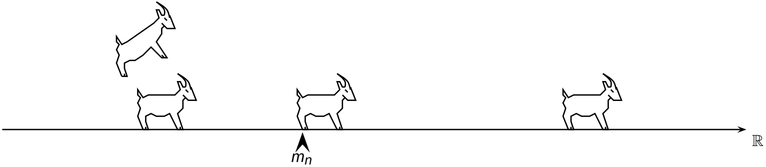
Goats jump on $\mathbb{R}$.



- n goats jump on $\mathbb{R}$ (state space is $\mathbb{R}^n$).
- Given a configuration $x_1, x_2, \ldots, x_n$ of goats the center of mass is $m_n = \frac{1}{n}\sum_{i=1}^{n} x_i$.
- Goat i jumps with rate $w(x_i - m_n)$, where w is the jump rate function: $\mathbb{R} \to \mathbb{R}^+$, decreasing.
- Jump lengths are positive, random, independent of everything, have mean one (and are of density φ).

The model (Bálint Tóth)

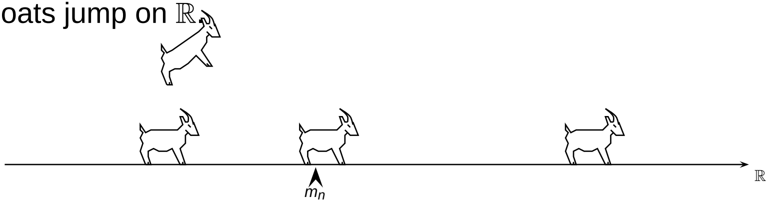
Goats jump on $\mathbb{R}$.



- n goats jump on $\mathbb{R}$ (state space is $\mathbb{R}^n$).
- Given a configuration $x_1, x_2, \ldots, x_n$ of goats the center of mass is $m_n = \frac{1}{n}\sum_{i=1}^{n} x_i$.
- Goat i jumps with rate $w(x_i - m_n)$, where w is the jump rate function: $\mathbb{R} \to \mathbb{R}^+$, decreasing.
- Jump lengths are positive, random, independent of everything, have mean one (and are of density φ).

The model (Bálint Tóth)

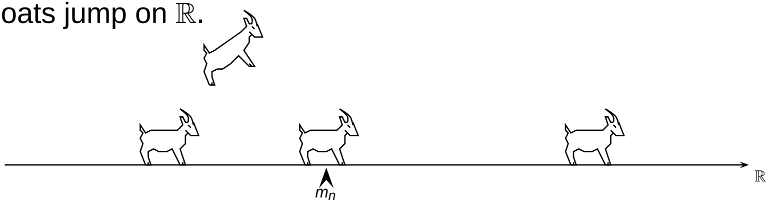
Goats jump on $\mathbb{R}$.



- n goats jump on $\mathbb{R}$ (state space is $\mathbb{R}^n$).
- Given a configuration $x_1, x_2, \ldots, x_n$ of goats the center of mass is $m_n = \frac{1}{n}\sum_{i=1}^n x_i$.
- Goat i jumps with rate $w(x_i - m_n)$, where w is the jump rate function: $\mathbb{R} \to \mathbb{R}^+$, decreasing.
- Jump lengths are positive, random, independent of everything, have mean one (and are of density φ).

The model (Bálint Tóth)

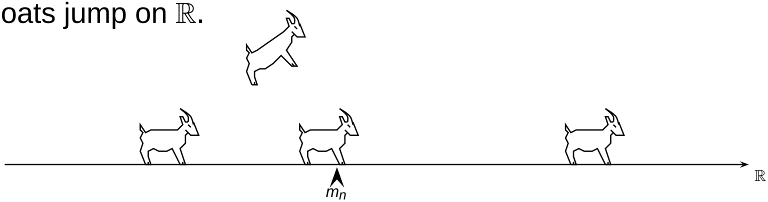
Goats jump on $\mathbb{R}$.



- n goats jump on $\mathbb{R}$ (state space is $\mathbb{R}^n$).
- Given a configuration $x_1, x_2, \ldots, x_n$ of goats the center of mass is $m_n = \frac{1}{n}\sum_{i=1}^{n} x_i$.
- Goat i jumps with rate $w(x_i - m_n)$, where w is the jump rate function: $\mathbb{R} \to \mathbb{R}^+$, decreasing.
- Jump lengths are positive, random, independent of everything, have mean one (and are of density φ).

The model (Bálint Tóth)

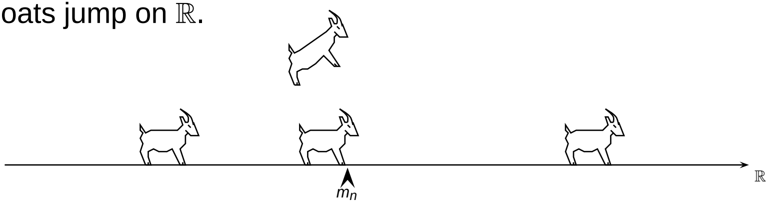
Goats jump on $\mathbb{R}$.



- n goats jump on $\mathbb{R}$ (state space is $\mathbb{R}^n$).
- Given a configuration $x_1, x_2, \ldots, x_n$ of goats the center of mass is $m_n = \frac{1}{n}\sum_{i=1}^{n} x_i$.
- Goat i jumps with rate $w(x_i - m_n)$, where w is the jump rate function: $\mathbb{R} \to \mathbb{R}^+$, decreasing.
- Jump lengths are positive, random, independent of everything, have mean one (and are of density φ).

The model (Bálint Tóth)

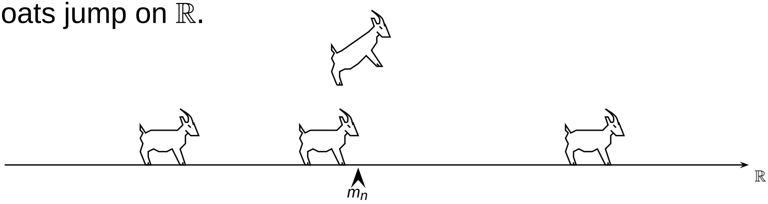
Goats jump on $\mathbb{R}$.



- n goats jump on $\mathbb{R}$ (state space is $\mathbb{R}^n$).
- Given a configuration $x_1, x_2, \ldots, x_n$ of goats the center of mass is $m_n = \frac{1}{n}\sum_{i=1}^{n} x_i$.
- Goat i jumps with rate $w(x_i - m_n)$, where w is the jump rate function: $\mathbb{R} \to \mathbb{R}^+$, decreasing.
- Jump lengths are positive, random, independent of everything, have mean one (and are of density φ).

The model (Bálint Tóth)

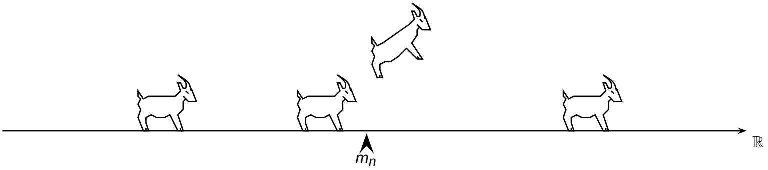
Goats jump on $\mathbb{R}$.



- n goats jump on $\mathbb{R}$ (state space is $\mathbb{R}^n$).
- Given a configuration $x_1, x_2, \ldots, x_n$ of goats the center of mass is $m_n = \frac{1}{n}\sum_{i=1}^{n} x_i$.
- Goat i jumps with rate $w(x_i - m_n)$, where w is the jump rate function: $\mathbb{R} \to \mathbb{R}^+$, decreasing.
- Jump lengths are positive, random, independent of everything, have mean one (and are of density φ).

The model (Bálint Tóth)

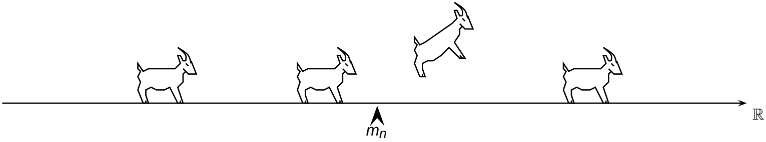
Goats jump on $\mathbb{R}$.



- n goats jump on $\mathbb{R}$ (state space is $\mathbb{R}^n$).
- Given a configuration $x_1, x_2, \ldots, x_n$ of goats the center of mass is $m_n = \frac{1}{n}\sum_{i=1}^n x_i$.
- Goat i jumps with rate $w(x_i - m_n)$, where w is the jump rate function: $\mathbb{R} \to \mathbb{R}^+$, decreasing.
- Jump lengths are positive, random, independent of everything, have mean one (and are of density φ).

The model (Bálint Tóth)

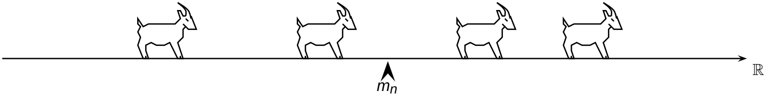
Goats jump on $\mathbb{R}$.



- n goats jump on $\mathbb{R}$ (state space is $\mathbb{R}^n$).
- Given a configuration $x_1, x_2, \ldots, x_n$ of goats the center of mass is $m_n = \frac{1}{n}\sum_{i=1}^{n} x_i$.
- Goat i jumps with rate $w(x_i - m_n)$, where w is the jump rate function: $\mathbb{R} \to \mathbb{R}^+$, decreasing.
- Jump lengths are positive, random, independent of everything, have mean one (and are of density φ).

The model (Bálint Tóth)

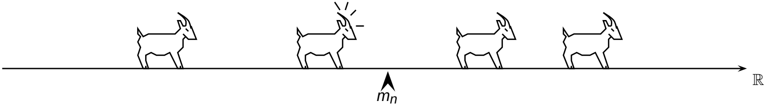
Goats jump on $\mathbb{R}$.



- n goats jump on $\mathbb{R}$ (state space is $\mathbb{R}^n$).
- Given a configuration $x_1, x_2, \ldots, x_n$ of goats the center of mass is $m_n = \frac{1}{n}\sum_{i=1}^{n} x_i$.
- Goat i jumps with rate $w(x_i - m_n)$, where w is the jump rate function: $\mathbb{R} \to \mathbb{R}^+$, decreasing.
- Jump lengths are positive, random, independent of everything, have mean one (and are of density φ).

The model (Bálint Tóth)

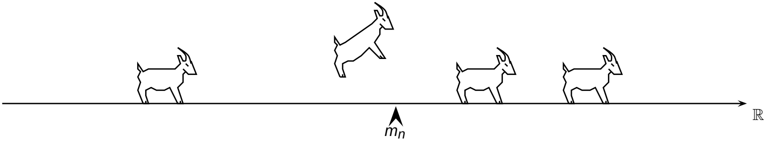
Goats jump on $\mathbb{R}$.



- n goats jump on $\mathbb{R}$ (state space is $\mathbb{R}^n$).
- Given a configuration $x_1, x_2, \ldots, x_n$ of goats the center of mass is $m_n = \frac{1}{n}\sum_{i=1}^n x_i$.
- Goat i jumps with rate $w(x_i - m_n)$, where w is the jump rate function: $\mathbb{R} \to \mathbb{R}^+$, decreasing.
- Jump lengths are positive, random, independent of everything, have mean one (and are of density φ).

The model (Bálint Tóth)

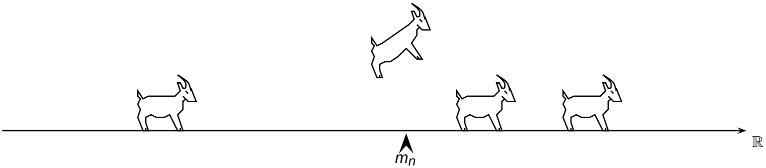
Goats jump on $\mathbb{R}$.



- n goats jump on $\mathbb{R}$ (state space is $\mathbb{R}^n$).
- Given a configuration $x_1, x_2, \ldots, x_n$ of goats the center of mass is $m_n = \frac{1}{n}\sum_{i=1}^{n} x_i$.
- Goat i jumps with rate $w(x_i - m_n)$, where w is the jump rate function: $\mathbb{R} \to \mathbb{R}^+$, decreasing.
- Jump lengths are positive, random, independent of everything, have mean one (and are of density φ).

The model (Bálint Tóth)

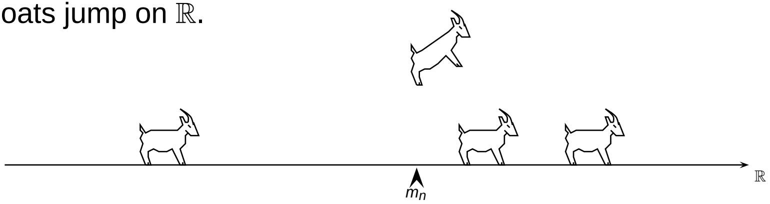
Goats jump on $\mathbb{R}$.



- n goats jump on $\mathbb{R}$ (state space is $\mathbb{R}^n$).
- Given a configuration $x_1, x_2, \ldots, x_n$ of goats the center of mass is $m_n = \frac{1}{n}\sum_{i=1}^n x_i$.
- Goat i jumps with rate $w(x_i - m_n)$, where w is the jump rate function: $\mathbb{R} \to \mathbb{R}^+$, decreasing.
- Jump lengths are positive, random, independent of everything, have mean one (and are of density φ).

The model (Bálint Tóth)

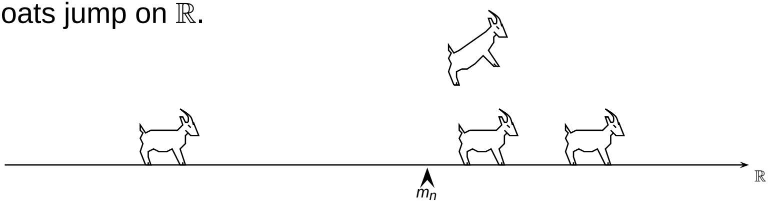
Goats jump on $\mathbb{R}$.



- n goats jump on $\mathbb{R}$ (state space is $\mathbb{R}^n$).
- Given a configuration $x_1, x_2, \dots, x_n$ of goats the center of mass is $m_n = \frac{1}{n}\sum_{i=1}^n x_i$.
- Goat i jumps with rate $w(x_i - m_n)$, where w is the jump rate function: $\mathbb{R} \to \mathbb{R}^+$, decreasing.
- Jump lengths are positive, random, independent of everything, have mean one (and are of density φ).

The model (Bálint Tóth)

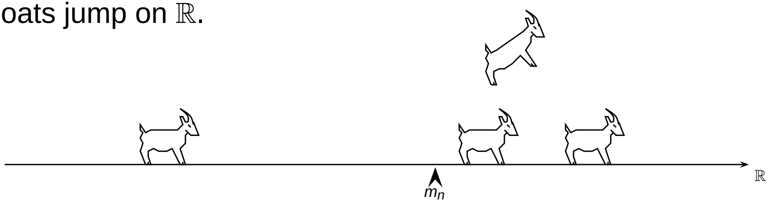
Goats jump on $\mathbb{R}$.



- n goats jump on $\mathbb{R}$ (state space is $\mathbb{R}^n$).
- Given a configuration $x_1, x_2, \ldots, x_n$ of goats the center of mass is $m_n = \frac{1}{n}\sum_{i=1}^{n} x_i$.
- Goat i jumps with rate $w(x_i - m_n)$, where w is the jump rate function: $\mathbb{R} \to \mathbb{R}^+$, decreasing.
- Jump lengths are positive, random, independent of everything, have mean one (and are of density φ).

The model (Bálint Tóth)

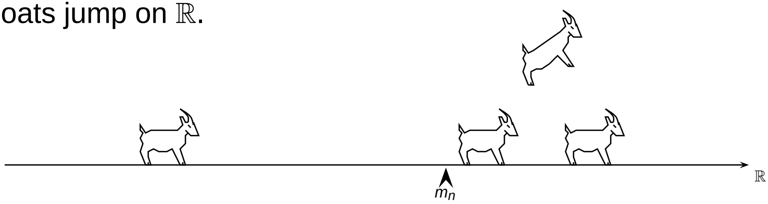
Goats jump on $\mathbb{R}$.



- n goats jump on $\mathbb{R}$ (state space is $\mathbb{R}^n$).
- Given a configuration $x_1, x_2, \ldots, x_n$ of goats the center of mass is $m_n = \frac{1}{n}\sum_{i=1}^n x_i$.
- Goat i jumps with rate $w(x_i - m_n)$, where w is the jump rate function: $\mathbb{R} \to \mathbb{R}^+$, decreasing.
- Jump lengths are positive, random, independent of everything, have mean one (and are of density φ).

The model (Bálint Tóth)

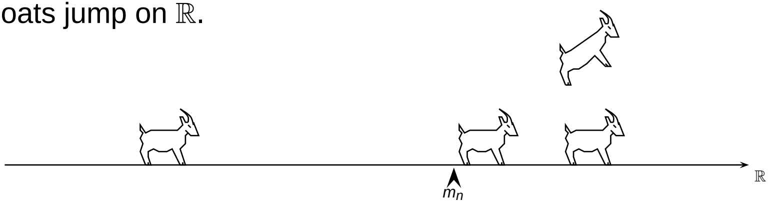
Goats jump on $\mathbb{R}$.



- n goats jump on $\mathbb{R}$ (state space is $\mathbb{R}^n$).
- Given a configuration $x_1, x_2, \ldots, x_n$ of goats the center of mass is $m_n = \frac{1}{n}\sum_{i=1}^{n} x_i$.
- Goat i jumps with rate $w(x_i - m_n)$, where w is the jump rate function: $\mathbb{R} \to \mathbb{R}^+$, decreasing.
- Jump lengths are positive, random, independent of everything, have mean one (and are of density φ).

The model (Bálint Tóth)

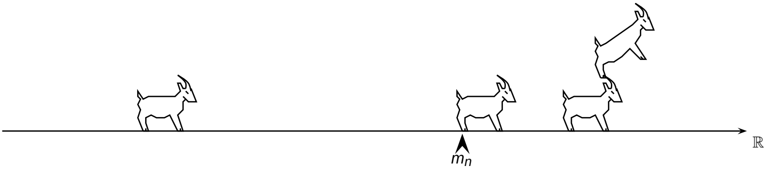
Goats jump on $\mathbb{R}$.



- n goats jump on $\mathbb{R}$ (state space is $\mathbb{R}^n$).
- Given a configuration $x_1, x_2, \ldots, x_n$ of goats the center of mass is $m_n = \frac{1}{n}\sum_{i=1}^n x_i$.
- Goat i jumps with rate $w(x_i - m_n)$, where w is the jump rate function: $\mathbb{R} \to \mathbb{R}^+$, decreasing.
- Jump lengths are positive, random, independent of everything, have mean one (and are of density φ).

The model (Bálint Tóth)

Goats jump on $\mathbb{R}$.



- n goats jump on $\mathbb{R}$ (state space is $\mathbb{R}^n$).
- Given a configuration $x_1, x_2, \ldots, x_n$ of goats the center of mass is $m_n = \frac{1}{n}\sum_{i=1}^n x_i$.
- Goat i jumps with rate $w(x_i - m_n)$, where w is the jump rate function: $\mathbb{R} \to \mathbb{R}^+$, decreasing.
- Jump lengths are positive, random, independent of everything, have mean one (and are of density φ).

The model (Bálint Tóth)

Goats jump on $\mathbb{R}$.

- n goats jump on $\mathbb{R}$ (state space is $\mathbb{R}^n$).
- Given a configuration $x_1, x_2, \ldots, x_n$ of goats the center of mass is $m_n = \frac{1}{n}\sum_{i=1}^{n} x_i$.
- Goat i jumps with rate $w(x_i - m_n)$, where w is the jump rate function: $\mathbb{R} \to \mathbb{R}^+$, decreasing.
- Jump lengths are positive, random, independent of everything, have mean one (and are of density φ).

The model (Bálint Tóth)

Goats jump on $\mathbb{R}$.

- n goats jump on $\mathbb{R}$ (state space is $\mathbb{R}^n$).
- Given a configuration $x_1, x_2, \ldots, x_n$ of goats the center of mass is $m_n = \frac{1}{n}\sum_{i=1}^n x_i$.
- Goat i jumps with rate $w(x_i - m_n)$, where w is the jump rate function: $\mathbb{R} \to \mathbb{R}^+$, decreasing.
- Jump lengths are positive, random, independent of everything, have mean one (and are of density φ).

The model (Bálint Tóth)

Goats jump on $\mathbb{R}$.

- n goats jump on $\mathbb{R}$ (state space is $\mathbb{R}^n$).
- Given a configuration $x_1, x_2, \ldots, x_n$ of goats the center of mass is $m_n = \frac{1}{n}\sum_{i=1}^{n} x_i$.
- Goat i jumps with rate $w(x_i - m_n)$, where w is the jump rate function: $\mathbb{R} \to \mathbb{R}^+$, decreasing.
- Jump lengths are positive, random, independent of everything, have mean one (and are of density φ).

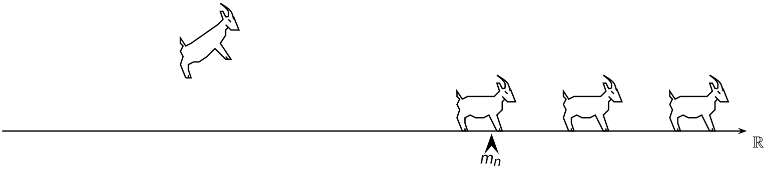
The model (Bálint Tóth)

Goats jump on $\mathbb{R}$.

- n goats jump on $\mathbb{R}$ (state space is $\mathbb{R}^n$).
- Given a configuration $x_1, x_2, \ldots, x_n$ of goats the center of mass is $m_n = \frac{1}{n}\sum_{i=1}^n x_i$.
- Goat i jumps with rate $w(x_i - m_n)$, where w is the jump rate function: $\mathbb{R} \to \mathbb{R}^+$, decreasing.
- Jump lengths are positive, random, independent of everything, have mean one (and are of density φ).

The model (Bálint Tóth)

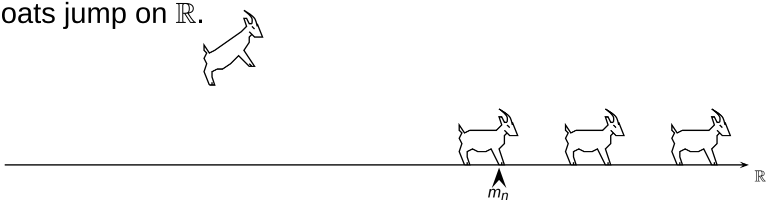
Goats jump on $\mathbb{R}$.



- n goats jump on $\mathbb{R}$ (state space is $\mathbb{R}^n$).
- Given a configuration $x_1, x_2, \ldots, x_n$ of goats the center of mass is $m_n = \frac{1}{n}\sum_{i=1}^n x_i$.
- Goat i jumps with rate $w(x_i - m_n)$, where w is the jump rate function: $\mathbb{R} \to \mathbb{R}^+$, decreasing.
- Jump lengths are positive, random, independent of everything, have mean one (and are of density φ).

The model (Bálint Tóth)

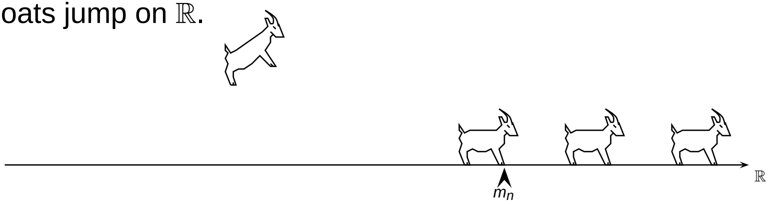
Goats jump on $\mathbb{R}$.



- n goats jump on $\mathbb{R}$ (state space is $\mathbb{R}^n$).
- Given a configuration $x_1, x_2, \ldots, x_n$ of goats the center of mass is $m_n = \frac{1}{n}\sum_{i=1}^n x_i$.
- Goat i jumps with rate $w(x_i - m_n)$, where w is the jump rate function: $\mathbb{R} \to \mathbb{R}^+$, decreasing.
- Jump lengths are positive, random, independent of everything, have mean one (and are of density φ).

The model (Bálint Tóth)

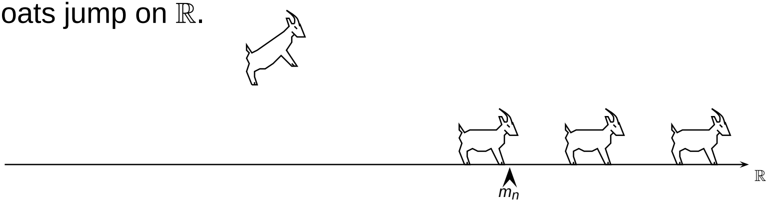
Goats jump on $\mathbb{R}$.



- n goats jump on $\mathbb{R}$ (state space is $\mathbb{R}^n$).
- Given a configuration $x_1, x_2, \dots, x_n$ of goats the center of mass is $m_n = \frac{1}{n}\sum_{i=1}^{n} x_i$.
- Goat i jumps with rate $w(x_i - m_n)$, where w is the jump rate function: $\mathbb{R} \to \mathbb{R}^+$, decreasing.
- Jump lengths are positive, random, independent of everything, have mean one (and are of density φ).

The model (Bálint Tóth)

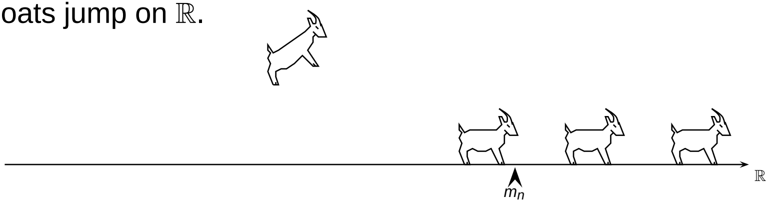
Goats jump on $\mathbb{R}$.



- n goats jump on $\mathbb{R}$ (state space is $\mathbb{R}^n$).
- Given a configuration $x_1, x_2, \ldots, x_n$ of goats the center of mass is $m_n = \frac{1}{n}\sum_{i=1}^{n} x_i$.
- Goat i jumps with rate $w(x_i - m_n)$, where w is the jump rate function: $\mathbb{R} \to \mathbb{R}^+$, decreasing.
- Jump lengths are positive, random, independent of everything, have mean one (and are of density φ).

The model (Bálint Tóth)

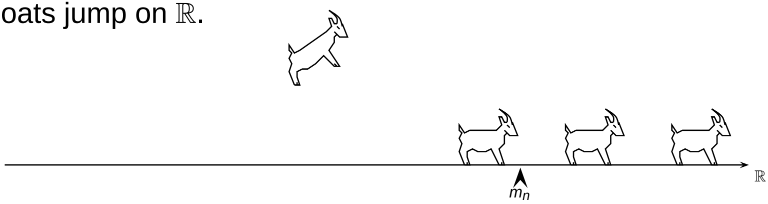
Goats jump on $\mathbb{R}$.



- n goats jump on $\mathbb{R}$ (state space is $\mathbb{R}^n$).
- Given a configuration $x_1, x_2, \ldots, x_n$ of goats the center of mass is $m_n = \frac{1}{n}\sum_{i=1}^{n} x_i$.
- Goat i jumps with rate $w(x_i - m_n)$, where w is the jump rate function: $\mathbb{R} \to \mathbb{R}^+$, decreasing.
- Jump lengths are positive, random, independent of everything, have mean one (and are of density φ).

The model (Bálint Tóth)

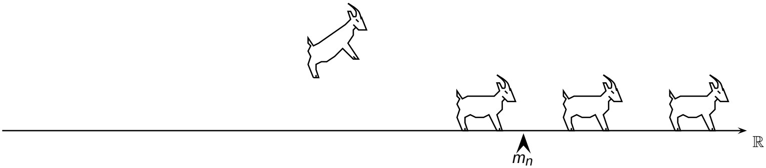
Goats jump on $\mathbb{R}$.



- n goats jump on $\mathbb{R}$ (state space is $\mathbb{R}^n$).
- Given a configuration $x_1, x_2, \ldots, x_n$ of goats the center of mass is $m_n = \frac{1}{n}\sum_{i=1}^n x_i$.
- Goat i jumps with rate $w(x_i - m_n)$, where w is the jump rate function: $\mathbb{R} \to \mathbb{R}^+$, decreasing.
- Jump lengths are positive, random, independent of everything, have mean one (and are of density φ).

The model (Bálint Tóth)

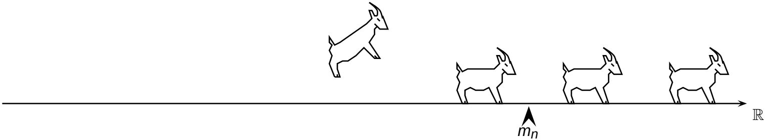
Goats jump on $\mathbb{R}$.



- n goats jump on $\mathbb{R}$ (state space is $\mathbb{R}^n$).
- Given a configuration $x_1, x_2, \ldots, x_n$ of goats the center of mass is $m_n = \frac{1}{n}\sum_{i=1}^n x_i$.
- Goat i jumps with rate $w(x_i - m_n)$, where w is the jump rate function: $\mathbb{R} \to \mathbb{R}^+$, decreasing.
- Jump lengths are positive, random, independent of everything, have mean one (and are of density φ).

The model (Bálint Tóth)

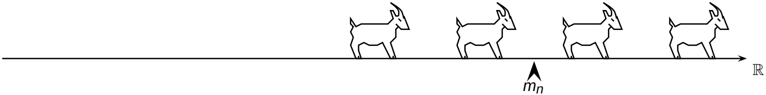
Goats jump on $\mathbb{R}$.



- n goats jump on $\mathbb{R}$ (state space is $\mathbb{R}^n$).
- Given a configuration $x_1, x_2, \ldots, x_n$ of goats the center of mass is $m_n = \frac{1}{n}\sum_{i=1}^n x_i$.
- Goat i jumps with rate $w(x_i - m_n)$, where w is the jump rate function: $\mathbb{R} \to \mathbb{R}^+$, decreasing.
- Jump lengths are positive, random, independent of everything, have mean one (and are of density φ).

The model (Bálint Tóth)

Goats jump on $\mathbb{R}$.



- n goats jump on $\mathbb{R}$ (state space is $\mathbb{R}^n$).
- Given a configuration $x_1, x_2, \ldots, x_n$ of goats the center of mass is $m_n = \frac{1}{n}\sum_{i=1}^n x_i$.
- Goat i jumps with rate $w(x_i - m_n)$, where w is the jump rate function: $\mathbb{R} \to \mathbb{R}^+$, decreasing.
- Jump lengths are positive, random, independent of everything, have mean one (and are of density φ).

The model

Can describe

- competing prices of goods (gyros / falafel / shawarma),

The model

Can describe

- competing prices of goods (gyros / falafel / shawarma),
- motion of flocks, herds (as you have seen...),

The model

Can describe

- competing prices of goods (gyros / falafel / shawarma),
- motion of flocks, herds (as you have seen...),
- etc.

The model

Can describe

- competing prices of goods (gyros / falafel / shawarma),
- motion of flocks, herds (as you have seen...),
- etc.

Similar models / results:

- Jump processes with interaction
  - ben-Avraham, Majumdar, Redner ’07
  - Greenberg, Malyshev, Popov ’95
  - Manita, Shcherbakov ’05
  - Grigorescu, Kang ’10
- Interacting diffusions with applications in stochastic portfolio theory
  - Banner, Fernholz, Karatzas ’04
  - Pal, Pitman ’08
  - Chatterjee, Pal ’10
  - Shkolnikov ’10, ’11, ’12

Stationary distribution

First question: what is the stationary distribution of the particles?

Stationary distribution

First question: what is the stationary distribution of the particles? As seen from the center of mass $m_n(t)$, of course.

Stationary distribution

First question: what is the stationary distribution of the particles? As seen from the center of mass $m_n(t)$, of course.

$n = 2$ particles: not hard.

Stationary distribution

First question: what is the stationary distribution of the particles? As seen from the center of mass $m_n(t)$, of course.

$n = 2$ particles: not hard.
(e.g. case $\varphi \sim$Exp(1) jumps, $w(x) = \mathrm{e}^{-2x}$ jump rates
$\rightsquigarrow$ logistic distribution , with density $\cosh^{-2}(z)$)

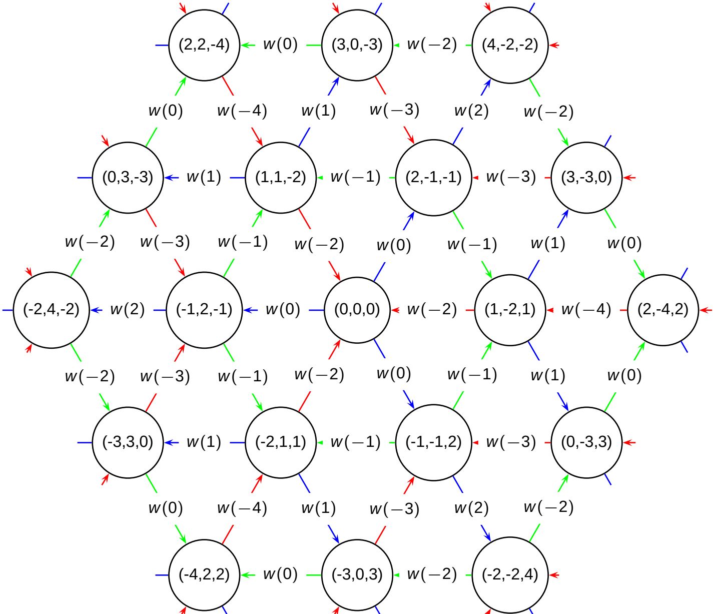
Stationary distribution

First question: what is the stationary distribution of the particles? As seen from the center of mass $m_n(t)$, of course.

$n = 2$ particles: not hard.
(e.g. case $\varphi \sim$Exp(1) jumps, $w(x) = e^{-2x}$ jump rates
$\rightsquigarrow$ logistic distribution , with density $\cosh^{-2}(z)$)

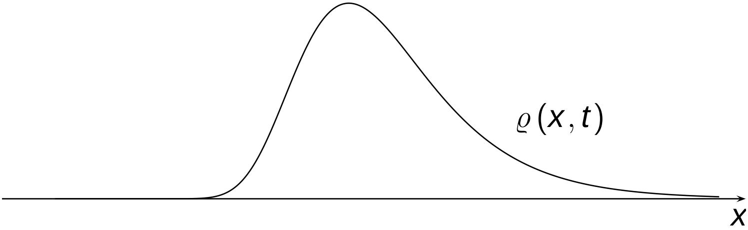
$n = 3$ particles: already seems hopeless. The process is "very irreversible".

$n = 3$ particles, jump lengths are deterministically 1



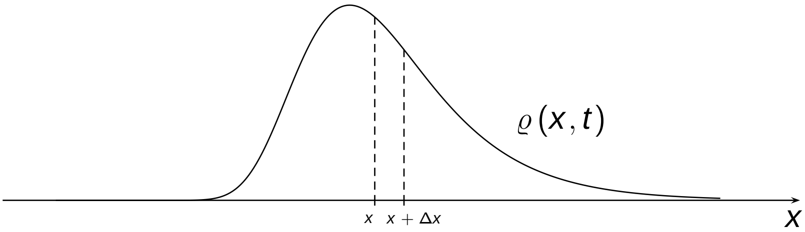
Mean field

- Mean field theory
  - Replaces all interactions with an average interaction
  - Many-particle problem $\rightsquigarrow$ one-particle problem
  - Resolves combinatorial problems
- $\varrho(x,t)$: probability density of position of particle at time t



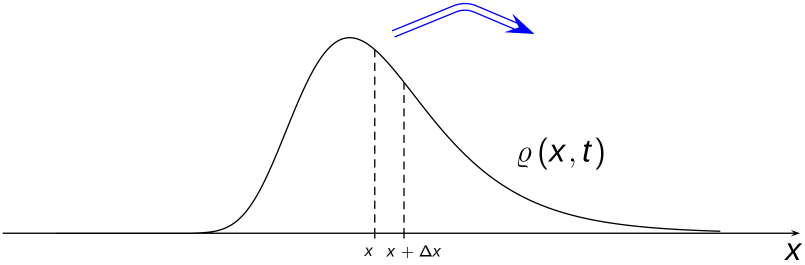
- Jump length is a random number from a probability distribution with density φ

Time evolution of $\varrho(x,t)$?



$$\frac{\partial \varrho(x,t)}{\partial t} = ???$$

Time evolution of $\varrho(x,t)$?

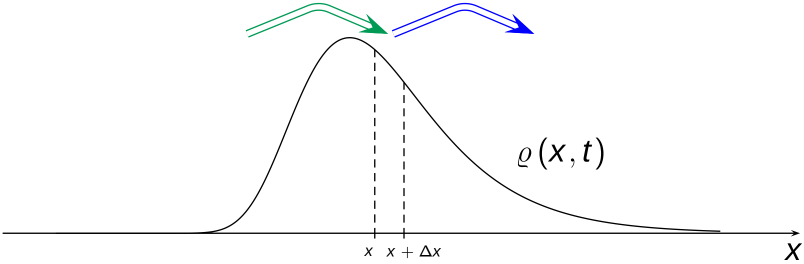


$$\frac{\partial \varrho(x,t)}{\partial t} = -w(x-m(t))\,\varrho(x,t) + \dots$$

where

$$m(t) = \int x\,\varrho(x,t)\,\mathrm{d}x.$$

Time evolution of $\varrho(x,t)$?



$$\frac{\partial \varrho(x,t)}{\partial t} = -w(x-m(t))\,\varrho(x,t) + \int_{-\infty}^{x} w(y-m(t))\,\varrho(y,t)\,\varphi(x-y)\,\mathrm{d}y$$

where

$$m(t) = \int x\,\varrho(x,t)\,\mathrm{d}x.$$

This is the mean field equation.

Stationary distribution as a travelling wave

- Stationary distribution as a travelling wave:

$$\varrho(x,t) = \rho(x - ct)$$

ρ: stationary distribution around the mean position.
c: speed of the wave.
ct: position of the mean.

- Mean field equation:

$$-c\rho'(x) = -w(x)\rho(x) + \int_{-\infty}^{x} w(y)\rho(y)\varphi(x-y)\,\mathrm{d}y.$$

Stationary distribution as a travelling wave

$$-c\rho'(x) = -w(x)\rho(x) + \int_{-\infty}^{x} w(y)\rho(y)\varphi(x-y)\,\mathrm{d}y.$$

Stationary distribution as a travelling wave

$$-c\rho'(x) = -w(x)\rho(x) + \int_{-\infty}^{x} w(y)\rho(y)\varphi(x-y)\,\mathrm{d}y.$$

Cases we can solve:

Stationary distribution as a travelling wave

$$-c\rho'(x) = -w(x)\rho(x) + \int_{-\infty}^{x} w(y)\rho(y)\varphi(x-y)\,\mathrm{d}y.$$

Cases we can solve:

- When the jumps are Exp(1): $\varphi(x) = \mathrm{e}^{-x}$, the above becomes a linear second order ODE, easy to solve.

Stationary distribution as a travelling wave

$$-c\rho'(x) = -w(x)\rho(x) + \int_{-\infty}^{x} w(y)\rho(y)\varphi(x-y)\,\mathrm{d}y.$$

Cases we can solve:

- When the jumps are Exp(1): $\varphi(x) = \mathrm{e}^{-x}$, the above becomes a linear second order ODE, easy to solve.
  - When $w(x) = \mathrm{e}^{-\beta x}$,

$$\rho(x) = G_{\frac{1}{\beta}}(\text{const}\cdot x),$$

$G_{\frac{1}{\beta}}$ is the generalized Gumbel density.

Stationary distribution as a travelling wave

$$-c\rho'(x) = -w(x)\rho(x) + \int_{-\infty}^{x} w(y)\rho(y)\varphi(x-y)\,\mathrm{d}y.$$

Cases we can solve:

- When the jumps are Exp(1): $\varphi(x) = \mathrm{e}^{-x}$, the above becomes a linear second order ODE, easy to solve.
  - When $w(x) = \mathrm{e}^{-\beta x}$,

    $$\rho(x) = G_{\frac{1}{\beta}}(\text{const}\cdot x),$$

    $G_{\frac{1}{\beta}}$ is the generalized Gumbel density.
  - When w is a (down-)step function, ρ is the Laplace density.

Stationary distribution as a travelling wave

$$-c\rho'(x) = -w(x)\rho(x) + \int_{-\infty}^{x} w(y)\rho(y)\varphi(x-y)\,\mathrm{d}y.$$

Cases we can solve:

- When the jumps are Exp(1): $\varphi(x) = \mathrm{e}^{-x}$, the above becomes a linear second order ODE, easy to solve.
  - When $w(x) = \mathrm{e}^{-\beta x}$,

    $$\rho(x) = G_{\frac{1}{\beta}}(\text{const}\cdot x),$$

    $G_{\frac{1}{\beta}}$ is the generalized Gumbel density.
  - When w is a (down-)step function, ρ is the Laplace density.
  - When w is a (down-)step function, but with a linear decrease around 0, ρ is Laplace with a normal segment in the middle.

Stationary distribution as a travelling wave

$$-c\rho'(x) = -w(x)\rho(x) + \int_{-\infty}^{x} w(y)\rho(y)\varphi(x-y)\,\mathrm{d}y.$$

Cases we can solve:

- When the jumps are Exp(1): $\varphi(x) = \mathrm{e}^{-x}$, the above becomes a linear second order ODE, easy to solve.
  - When $w(x) = \mathrm{e}^{-\beta x}$,

    $$\rho(x) = G_{\frac{1}{\beta}}(\mathrm{const} \cdot x),$$

    $G_{\frac{1}{\beta}}$ is the generalized Gumbel density.
  - When w is a (down-)step function, ρ is the Laplace density.
  - When w is a (down-)step function, but with a linear decrease around 0, ρ is Laplace with a normal segment in the middle.

Extreme value statistics (Attila Rákos)

When the jumps are Exp(1): $\varphi(x) = \mathrm{e}^{-x}$,
jump rate is exponential: $w(x) = \mathrm{e}^{-x}$,
$\rightsquigarrow \rho(x) = G(x - \gamma)$, centered std. Gumbel density. Why?

Extreme value statistics (Attila Rákos)

When the jumps are Exp(1): $\varphi(x) = \mathrm{e}^{-x}$,
jump rate is exponential: $w(x) = \mathrm{e}^{-x}$,
$\rightsquigarrow \rho(x) = G(x - \gamma)$, centered std. Gumbel density. Why?

Mean field particle $X(t)$ jumps with rate approx. $\mathrm{e}^{ct - X(t)}$. And when it jumps, it jumps Exp(1).

Extreme value statistics (Attila Rákos)

When the jumps are Exp(1): $\varphi(x) = \mathrm{e}^{-x}$,
jump rate is exponential: $w(x) = \mathrm{e}^{-x}$,
$\rightsquigarrow \rho(x) = G(x - \gamma)$, centered std. Gumbel density. Why?

Mean field particle $X(t)$ jumps with rate approx. $\mathrm{e}^{ct - X(t)}$. And when it jumps, it jumps Exp(1).

Extreme value statistics (Attila Rákos)

When the jumps are Exp(1): $\varphi(x) = \mathrm{e}^{-x}$,
jump rate is exponential: $w(x) = \mathrm{e}^{-x}$,
$\rightsquigarrow$ $\rho(x) = G(x - \gamma)$, centered std. Gumbel density. Why?

Mean field particle $X(t)$ jumps with rate approx. $\mathrm{e}^{ct - X(t)}$. And when it jumps, it jumps Exp(1).

Take now more and more iid. Exp(1) variables. At time t, let us have $N(t) = \mathrm{e}^{ct}/c$ of them. Define $Y(t)$ as the maximum.

Extreme value statistics (Attila Rákos)

When the jumps are Exp(1): $\varphi(x) = \mathrm{e}^{-x}$,
jump rate is exponential: $w(x) = \mathrm{e}^{-x}$,
$\rightsquigarrow \rho(x) = G(x - \gamma)$, centered std. Gumbel density. Why?

Mean field particle $X(t)$ jumps with rate approx. $\mathrm{e}^{ct - X(t)}$. And when it jumps, it jumps Exp(1).

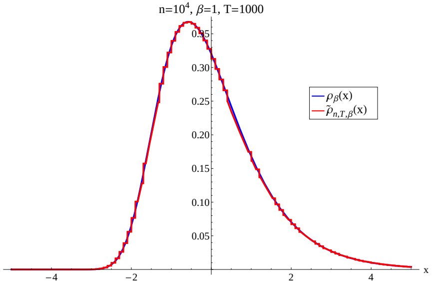
Take now more and more iid. Exp(1) variables. At time t, let us have $N(t) = \mathrm{e}^{ct}/c$ of them. Define $Y(t)$ as the maximum.

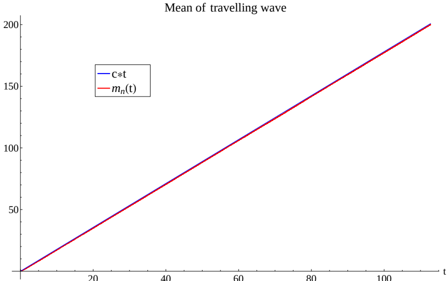
Between t and $t + \mathrm{d}t$, $\mathrm{d}N(t) = \mathrm{e}^{ct}\,\mathrm{d}t$ many new Exp(1) variables try to break the record. So the probability that $Y(t)$ jumps is

$$1 - \left(1 - \mathrm{e}^{-Y(t)}\right)^{\mathrm{e}^{ct}\,\mathrm{d}t} \simeq \mathrm{e}^{ct - Y(t)}\,\mathrm{d}t \qquad \text{(for large } Y(t)\text{)}.$$

And when it jumps, it jumps Exp(1). But we know that $Y(t) - ct + \log c$ converges to standard Gumbel.

Mean field is a good approximation





Fluid limit

Recall the original mean field equation:

$$\frac{\partial \varrho(x,t)}{\partial t} = -w(x - m(t)) \cdot \varrho(x,t) + \int_{-\infty}^{x} w(y - m(t)) \cdot \varrho(y,t) \cdot \varphi(x-y)\, \mathrm{d}y,$$

or, for all f test functions:

$$\langle f, \mu(t)\rangle - \langle f, \mu(0)\rangle = \int_0^t \Big\langle \big\{ \mathbf{E}[f(x+Z)] - f(x) \big\} w(x - m(s)),\ \mu(s) \Big\rangle\, \mathrm{d}s,$$

$$m(s) = \langle x,\ \mu(s)\rangle.$$

Here $\mathbf{E}$ refers to expectation of Z w.r.t. the jump length distribution.

Fluid limit

Goal: mean field equation holds in the fluid limit.

Fluid limit

Goal: mean field equation holds in the fluid limit.

Empirical measure: $\mu_n(t) = \frac{1}{n} \sum_{i=1}^{n} \delta_{x_i(t)}$.

Fluid limit

Goal: mean field equation holds in the fluid limit.

Empirical measure: $\mu_n(t) = \frac{1}{n}\sum_{i=1}^{n} \delta_{x_i(t)}$.

As a process: $\mu_n(\cdot) \in D([0,\infty), \mathcal{P}_1(\mathbb{R}))$.

Fluid limit

Goal: mean field equation holds in the fluid limit.

Empirical measure: $\mu_n(t) = \frac{1}{n}\sum_{i=1}^{n} \delta_{x_i(t)}$.

As a process: $\mu_n(\cdot) \in D([0,\infty), \mathcal{P}_1(\mathbb{R}))$.

Theorem

Assume that

- *w is bounded and [..];*
- *Z has a finite third moment;*
- *$\mu_n(0) \Rightarrow_n \nu$ in $\mathcal{P}_1(\mathbb{R})$, where ν is a deterministic measure, and [..].*

Then

$$\mu_n(\cdot) \Rightarrow_n \mu(\cdot)$$

in $D([0,\infty), \mathcal{P}_1(\mathbb{R}))$, where $\mu(\cdot)$ is the unique deterministic solution to the MFE with initial condition $\mu(0) = \nu$.

Fluid limit – proof outline

By general theory, we need to show three things:

- Tightness: $\{\mu_n(\cdot)\}_{n\geq 1}$ is tight in $D([0,\infty),\mathcal{P}_1(\mathbb{R}))$.
- Identification of the limit: any weak limit $\mu(\cdot)$ solves the MFE.
- Uniqueness: the MFE with a given initial condition has a unique solution.

Fluid limit – proof outline

By general theory, we need to show three things:

- Tightness: $\{\mu_n(\cdot)\}_{n\geq 1}$ is tight in $D([0,\infty),\mathcal{P}_1(\mathbb{R}))$.
- Identification of the limit: any weak limit $\mu(\cdot)$ solves the MFE.
- Uniqueness: the MFE with a given initial condition has a unique solution.

Note: working on $M_1(\mathbb{R})$ with topology of weak convergence is not enough.
Work on $\mathcal{P}_1(\mathbb{R})$ with Wasserstein-1 metric $\rightsquigarrow$ convergence of mean.

Where do we live?

Wasserstein metric on $\mathcal{P}_1(\mathbb{R})$:

$$d_1(\mu,\nu) = \inf_{\pi:\text{coupling meas.}} \int_{\mathbb{R}\times\mathbb{R}} |x-y|\,\pi(dx,dy).$$

Test functions:

$$H := \{f : f \in C_b, |f| \leq 1\} \cup \{\mathrm{Id}\}.$$

Convergence in d_1 implies convergence of the integrals of such test functions.

All these needed to be able to handle the center of mass

$$m(s) = \langle x, \mu(s)\rangle.$$

Tightness

- Step 1: Tightness of $\langle f, \mu_n(\cdot)\rangle$ in $D([0,\infty), \mathbb{R})$ for f bounded and continuous, and also for $f = \mathrm{Id}$.
  - Need uniform control of tails at time zero (just assume these),
  - uniform control of jumps (Billingsley's book).
- Step 2: Any limit point is a.s. continuous.
  - Further conditions on jumps (Ethier & Kurtz book).

$\Big\}$ *C-relative compactness*

Method for these bounds: introduce *ghost goats*: they jump with rate $\sup_x w(x)$, they have the same jump length distribution as their original counterparts. Couple such that ghost goat$_i$ can jump without goat$_i$, but not vice-versa. $\rightsquigarrow$ increments of ghosts dominate increments of the original goats.

Tightness

- Step 3: C-relative compactness of $\mu_n(\cdot)$ in $D([0,\infty),\mathcal{P}_1(\mathbb{R}))$.
  - Check compactness-type condition for $\mu_n(t)$, uniformly in n and t,
  - C-relative compactness of $\langle f, \mu_n(\cdot)\rangle$ in $D([0,\infty),\mathbb{R})$ from previous slide.
  - Generalize Perkins's theorem (Perkins, St.-Flour notes, 1999).

For compactness-type condition, use again the ghost goats.

Perkins's theorem originally was about checking C-relative compactness in $D([0,\infty),\mathcal{M})$ by checking that of appropriate integrals $\langle f, \mu_n(\cdot)\rangle$ in $D([0,\infty),\mathbb{R})$. Our job here was to slightly generalize from finite measures $\mathcal{M}$ to measures with finite first moment $\mathcal{P}_1$.

Any limit solves the mean field equation

Let

$$\begin{aligned} A_{t,f}(\mu(\cdot)) :&= \langle f, \mu(t)\rangle - \langle f, \mu(0)\rangle \\ &\quad - \int_0^t \Big\langle \Big\{ \mathbf{E}[f(x+Z)] - f(x) \Big\} w(x - m(s)),\ \mu(s) \Big\rangle ds \\ &= \langle f, \mu(t)\rangle - \langle f, \mu(0)\rangle - \int_0^t L\langle f, \mu(s)\rangle ds, \end{aligned}$$

where

$$m(s) = \langle x, \mu(s)\rangle.$$

Recall that the mean field equation is

$$A_{t,f}(\mu(\cdot)) = 0$$

for all $t \geq 0$ and test functions $f \in H$.

Any limit solves the mean field equation

- Step 1:
$$\sup_{0\le s\le t}\left|A_{s,f}\left(\mu_n\left(\cdot\right)\right)\right| \xrightarrow[n\to\infty]{\mathbb{P}} 0.$$
- Step 2: If $\mu_n\left(\cdot\right) \Rightarrow_n \mu\left(\cdot\right)$ in $D\left(\left[0,\infty\right),\mathcal{P}_1\left(\mathbb{R}\right)\right)$, then for every $t\geq 0$ and every $f\in H$,
$$A_{t,f}\left(\mu_n\left(\cdot\right)\right)\Rightarrow_n A_{t,f}\left(\mu\left(\cdot\right)\right)$$
in $\mathbb{R}$.

Any limit solves the mean field equation

- Step 1:
$$\sup_{0\le s\le t} \left|A_{s,f}\left(\mu_n\left(\cdot\right)\right)\right| \xrightarrow[n\to\infty]{\mathbb{P}} 0.$$
- Step 2: If $\mu_n\left(\cdot\right) \Rightarrow_n \mu\left(\cdot\right)$ in $D\left(\left[0,\infty\right),\mathcal{P}_1\left(\mathbb{R}\right)\right)$, then for every $t \geq 0$ and every $f \in H$,
$$A_{t,f}\left(\mu_n\left(\cdot\right)\right) \Rightarrow_n A_{t,f}\left(\mu\left(\cdot\right)\right)$$
in $\mathbb{R}$.

For the first step, notice that $A_{t,f}\left(\mu_n\left(\cdot\right)\right)$ is a martingale in t. Use Doob's inequality and show that the L^2 norm goes to zero as $n \to \infty$.

Any limit solves the mean field equation

- Step 1:

$$\sup_{0\le s\le t} \left| A_{s,f}\left(\mu_n\left(\cdot\right)\right) \right| \xrightarrow[n\to\infty]{\mathbb{P}} 0.$$

- Step 2: If $\mu_n(\cdot) \Rightarrow_n \mu(\cdot)$ in $D([0,\infty), \mathcal{P}_1(\mathbb{R}))$, then for every $t \ge 0$ and every $f \in H$,

$$A_{t,f}\left(\mu_n\left(\cdot\right)\right) \Rightarrow_n A_{t,f}\left(\mu\left(\cdot\right)\right)$$

in $\mathbb{R}$.

For the first step, notice that $A_{t,f}(\mu_n(\cdot))$ is a martingale in t. Use Doob's inequality and show that the L^2 norm goes to zero as $n \to \infty$.

For the second step, convergence in $D([0,\infty), \mathcal{P}_1(\mathbb{R}))$ with the Wasserstein metric d_1 is just right for our test functions (including the center of mass!).

Uniqueness of solutions of the mean field equation

- Step 1: Look at the distance

$$d_H(\mu,\nu) := \sup_{f \in H} |\langle f, \mu\rangle - \langle f, \nu\rangle| .$$

- Step 2: Apply to solutions $\mu(\cdot)$ and $\nu(\cdot)$ of the mean field equation:

$$\langle f, \mu(t)\rangle = \langle f, \mu(0)\rangle + \int_0^t \left\langle \left\{ \mathbf{E}[f(x+Z)] - f(x) \right\} w(x - m(s)),\ \mu(s) \right\rangle ds.$$

Terms in the difference of integrals can be bounded in terms of $d_H(\mu(s), \nu(s))$.

$\rightsquigarrow d_H(\mu(t), \nu(t)) \le d_H(\mu(0), \nu(0)) + c \int_0^t d_H(\mu(s), \nu(s))\, ds$, apply Grönwall's inequality.

Summary

Our main contributions are

- introducing the model;
- formulating via heuristics and analyzing the limiting behavior as $n \to \infty$;
- and showing rigorously that (under some assumptions) in the fluid limit the process indeed satisfies the deterministic integro-differential equation that we formulated.

Questions

- Fluctuations: variance of the center of mass should scale:

$$\mathbf{Var}(m_n(t)) \sim \frac{t^\gamma}{n^\alpha}.$$

Some (small) simulations suggest that:

Questions

- Fluctuations: variance of the center of mass should scale:

$$\mathbf{Var}(m_n(t)) \sim \frac{t^\gamma}{n^\alpha}.$$

Some (small) simulations suggest that:

- Exponential jump rates, exponential jumps: $\gamma \simeq \alpha \simeq 1$.

Questions

- Fluctuations: variance of the center of mass should scale:

$$\mathbf{Var}(m_n(t)) \sim \frac{t^\gamma}{n^\alpha}.$$

Some (small) simulations suggest that:

  - Exponential jump rates, exponential jumps: $\gamma \simeq \alpha \simeq 1$.
  - Step function jump rates, exponential jumps: $\gamma \simeq 1,\ 1/2 \leq \alpha \leq 1$.

Questions

- Fluctuations: variance of the center of mass should scale:

$$\mathbf{Var}(m_n(t)) \sim \frac{t^\gamma}{n^\alpha}.$$

  Some (small) simulations suggest that:
  - Exponential jump rates, exponential jumps: $\gamma \simeq \alpha \simeq 1$.
  - Step function jump rates, exponential jumps: $\gamma \simeq 1,\ 1/2 \leq \alpha \leq 1$.
  - Step function with linear segment jump rates, exponential jumps: $\gamma \simeq 1,\ 1/2 \leq \alpha \leq 1$.

Questions

- Fluctuations: variance of the center of mass should scale:

$$\mathbf{Var}(m_n(t)) \sim \frac{t^\gamma}{n^\alpha}.$$

  Some (small) simulations suggest that:
  - Exponential jump rates, exponential jumps: $\gamma \simeq \alpha \simeq 1$.
  - Step function jump rates, exponential jumps: $\gamma \simeq 1,\ 1/2 \leq \alpha \leq 1$.
  - Step function with linear segment jump rates, exponential jumps: $\gamma \simeq 1,\ 1/2 \leq \alpha \leq 1$.
- In general, limit distribution theorems?

Questions

- Fluctuations: variance of the center of mass should scale:

$$\mathbf{Var}(m_n(t)) \sim \frac{t^\gamma}{n^\alpha}.$$

  Some (small) simulations suggest that:
  - Exponential jump rates, exponential jumps: $\gamma \simeq \alpha \simeq 1$.
  - Step function jump rates, exponential jumps: $\gamma \simeq 1,\ 1/2 \leq \alpha \leq 1$.
  - Step function with linear segment jump rates, exponential jumps: $\gamma \simeq 1,\ 1/2 \leq \alpha \leq 1$.
- In general, limit distribution theorems?
- Can we really not find the stationary distribution for three goats?

Questions

- Fluctuations: variance of the center of mass should scale:

$$\mathbf{Var}(m_n(t)) \sim \frac{t^\gamma}{n^\alpha}.$$

  Some (small) simulations suggest that:
  - Exponential jump rates, exponential jumps: $\gamma \simeq \alpha \simeq 1$.
  - Step function jump rates, exponential jumps: $\gamma \simeq 1,\ 1/2 \leq \alpha \leq 1$.
  - Step function with linear segment jump rates, exponential jumps: $\gamma \simeq 1,\ 1/2 \leq \alpha \leq 1$.
- In general, limit distribution theorems?
- Can we really not find the stationary distribution for three goats?
- And for the fluid limit, general rate functions / jump distributions?

Questions

- Fluctuations: variance of the center of mass should scale:

$$\mathbf{Var}(m_n(t)) \sim \frac{t^\gamma}{n^\alpha}.$$

  Some (small) simulations suggest that:
  - Exponential jump rates, exponential jumps: $\gamma \simeq \alpha \simeq 1$.
  - Step function jump rates, exponential jumps: $\gamma \simeq 1,\ 1/2 \leq \alpha \leq 1$.
  - Step function with linear segment jump rates, exponential jumps: $\gamma \simeq 1,\ 1/2 \leq \alpha \leq 1$.
- In general, limit distribution theorems?
- Can we really not find the stationary distribution for three goats?
- And for the fluid limit, general rate functions / jump distributions?

Thank you!