Braess's paradox for the spectral gap in random graphs

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Overview

How do connectivity/spectral properties of a random graph change when adding an edge?
We show that a

constant fraction of nonedges

are such that their addition

decreases the spectral gap.

The result hinges on a
new delocalization result for the 2^{nd} eigenvector.

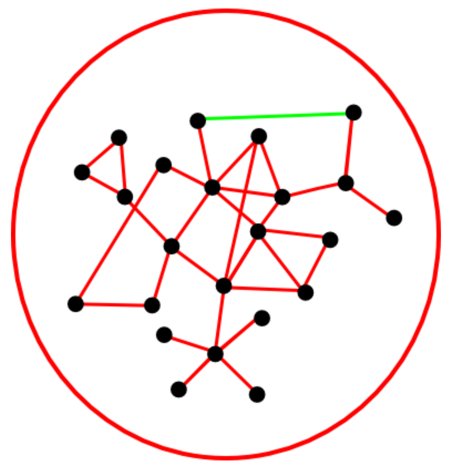
Braess's paradox

The addition of an extra road in a traffic network can increase the overall journey time for all drivers.

Adding an edge to random graphs

$G \sim G(n,p)$, $p \in (0,1)$ fixed
$G_+ = G \cup \{$ an additional edge $\}$

How do connectivity/spectral properties change?



Symmetric normalized Laplacian:

$$\mathcal{L}_G = D^{-1/2}(D-A)D^{-1/2},$$

where A is the adjacency matrix and D is the diagonal degree matrix. $\mathcal{L}_G$ has eigenvalues:

$$0 = \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \le 2.$$

Interested in λ_2, known as the spectral gap.

- $\lambda_2 = 0$ iff G is disconnected
- $1/\lambda_2$ is the relaxation time of the simple RW on G

Conjecture (Fan Chung, 2014).
A constant fraction of edges of $G(n,p)$ are such that their removal increases the spectral gap.

Main result

For a graph $G = (V,E)$, let

$a_-(G) :=$ fraction of nonedges in G whose addition decreases λ_2,

i.e., for which

$$\lambda_2(\mathcal{L}_{G_+}) < \lambda_2(\mathcal{L}_G).$$

Theorem. Let $p \in (0,1)$ be fixed. There exists a constant $c = c(p) > 0$ such that

$$\mathbb{P}\left[a_-(G(n,p)) \ge c\right] \to 1.$$

Moreover, one can take $c = 1/8 - \delta$ for any constant $\delta > 0$.

Comments:

- **The optimal constant** should be $c = 1/2 - \delta$. See below for more.
- **Normalization is key** to this phenomenon. For the combinatorial Laplacian $L = D - A$ adding an edge always increases the spectral gap.
- **Removing an edge:** the results should be similar, but the proof of the one above is asymmetric. See below for more.
- **Sparse graphs:** Our proofs show that the results hold for $p = n^{-\varepsilon}$ for some $\varepsilon > 0$. We did not try to optimize the dependence on p.

Proof ideas

The first eigenvector of $\mathcal{L}_G$ is $D^{1/2}\mathbf{1}$. Let f_2 be the 2^{nd} eigenvector of $\mathcal{L}_G$.
Idea: consider f_2 as the 2^{nd} eigenvector of $\mathcal{L}_{G_+}$.

$$\lambda_2(\mathcal{L}_{G_+}) = \min_{x : x \perp D_+^{1/2}\mathbf{1}} \frac{x^T \mathcal{L}_{G_+} x}{x^T x} \lesssim \frac{f_2^T \mathcal{L}_{G_+} f_2}{f_2^T f_2}$$

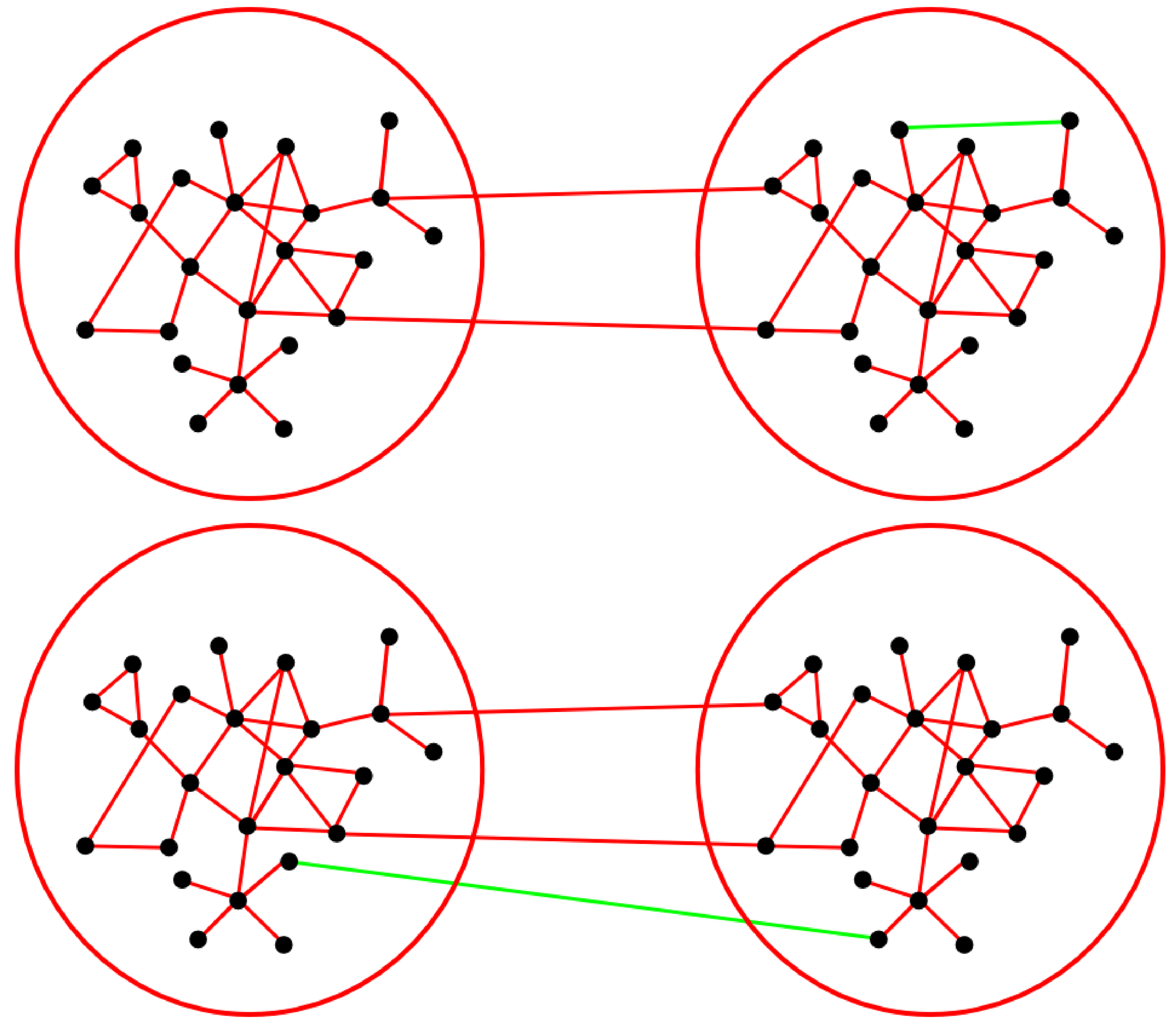
This gives a general sufficient condition for the spectral gap to decrease. Specialized to $G(n,p)$ this becomes

$$\frac{32}{\sqrt{np}}\left(f_2(u)^2 + f_2(v)^2\right) + \frac{8}{(np)^2} < f_2(u)\, f_2(v).$$

The result above then follows from the following delocalization result:
Theorem. Fix $p \in (0,1)$. Let f_2 be the 2^{nd} eigenvector of $\mathcal{L}_{G(n,p)}$ with unit norm. For every $\delta > 0$ there exists a constant $C = C(p,\delta)$ such that

$$\mathbb{P}\left[\frac{1}{n}\#\left\{i : |f_2(i)| \ge \frac{1}{\sqrt{n}\,(\log(n))^C}\right\} \ge \frac{1}{2} - \delta\right] \to 1.$$

Intuition



Future directions

- **Full delocalization.** This would imply the optimal constant in the main theorem.
- **Sparse graphs.** What happens when p goes to 0 with n?
- **Removing an edge.**
- **How do other notions of connectivity/mixing change?**

Acknowledgements

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[1] R. Eldan, M.Z. Rácz, T. Schramm. Braess's paradox for the spectral gap in random graphs and delocalization of eigenvectors. Preprint at http://arxiv.org/abs/1504.07669.