Multidimensional sticky Brownian motions as limits of exclusion processes

Joint work with Mykhaylo Shkolnikov

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Symmetric simple exclusion process

Particles try to jump
  ▶ with rate 1 to the left, and
  ▶ with rate 1 to the right.

Exclusion:
the jump is suppressed if the destination site is occupied.
Motivation: SPT  Reflected BM  Sticky BM  Our results

Symmetric simple exclusion with *stickiness*

When apart, particles jump as usual:

When there is a collision, the whole system slows down:
Symmetric simple exclusion with *stickiness*

When apart, particles jump as usual:

When there is a collision, the whole system slows down:

**Topic of today’s talk:**
Describing the *diffusive scaling limit* of such particle systems.
Outline

Motivation: stochastic portfolio theory

Queueing theory and reflected Brownian motion

Sticky Brownian motion in 1D

Our results
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Our results
log(market rank) vs. log(market capitalization)
for US stock markets in 1929, . . . , 2009

Question: why is this curve stable?
Stochastic portfolio theory: a descriptive theory

- Developed by Fernholz (2002), Banner, Fernholz, and Karatzas (2005), and coauthors.
- Rank-based models:

\[
\frac{d}{dt} \log X_i(t) = \gamma_{r_t(i)}(t) \, dt + \sigma_{r_t(i)}(t) \, dW_i(t),
\]

where \( r_t(i) \) is the rank of particle \( i \) at time \( t \).
- Can describe the observed capital distribution curves.

However, rank-based models have shortcomings. To address this:
- A second-order stock market model: rank- and name-based (Fernholz, Ichiba, Karatzas, 2013)
- Brownian particles with asymmetric collisions (Karatzas, Pal, Shkolnikov, 2012)

Motivation: models that describe events that slow down the market (e.g., a court trial)?
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Our results
A simple flow model

- $X_0$: initial inventory level,
- $A, B$: increasing, continuous stochastic processes,
- $A_t$: cumulative input until time $t$,
- $B_t$: cumulative potential output until time $t$,
- $L_t$: amount of potential output lost until time $t$,
- $X_t := X_0 + A_t - B_t$: netput process
- $Z_t := X_0 + A_t - (B_t - L_t) = X_t + L_t$: inventory process

Model: $X$ is a Brownian motion; reasonable for balanced, high volume flows. Then: $Z$ is a reflected Brownian motion, and $L$ is its local time process at zero.
Motivation: SPT  Reflected BM  Sticky BM  Our results

Reflected Brownian motion

Definition

Let $\eta \in \mathbb{R}^d$, let $\Gamma$ be a $d \times d$ non-degenerate covariance matrix, let $R$ be a $d \times d$ matrix. For $x \in (\mathbb{R}_+)^d$, a semimartingale reflecting Brownian motion (SRBM) in the orthant $(\mathbb{R}_+)^d$ associated with the data $(\eta, \Gamma, R)$ that starts from $x$ is a continuous, $(\mathcal{F}_t)$-adapted, $d$-dimensional process $Z$ s.t.

$$Z(t) = X(t) + RY(t) \in (\mathbb{R}_+)^d \quad \text{for all } t \geq 0,$$

(i) $X$ is a $d$-dimensional Brownian motion with drift vector $\eta$ and covariance matrix $\Gamma$ such that $\{X(t) - \eta t, \mathcal{F}_t, t \geq 0\}$ is a martingale and $X(0) = x$ $P_x$-a.s.,

(ii) $Y$ is an $(\mathcal{F}_t)$-adapted, $d$-dimensional process such that $P_x$-a.s. for each $i \in [d]$, the $i^{th}$ component $Y_i$ of $Y$ satisfies

(a) $Y_i(0) = 0$,

(b) $Y_i$ is continuous and nondecreasing,

(c) $Y_i$ can increase only when $Z$ is on the face $F_i$.

$Y$ is referred to as the “pushing” process of $Z$. 
Reflected Brownian motion

- Studied heavily in the ’80s: Harrison, Reiman, Williams et al.
- Plays a key role in understanding queues with high traffic.

Existence and uniqueness:

**Theorem (Taylor and Williams (1993))**

*There exists a SRBM in the orthant* $(\mathbb{R}_+)^d$ *with data* $(\eta, \Gamma, R)$ *that starts from* $x \in (\mathbb{R}_+)^d$ *if and only if* $R$ *is completely-$S$. Moreover, when it exists, the joint law of any SRBM, together with its associated pushing process, is unique.*

Intuitively: completely-$S$ condition necessary to stay in the orthant.

**Definition (Completely-$S$ matrix)**

A matrix $d \times d$ matrix $A$ is *completely-$S$* if there exists $\lambda \in [0, \infty)^d$ such that $A\lambda \in (0, \infty)^d$, and the same property is shared by every principal submatrix of $A$. 
Reflected Brownian motion

No time on the boundary:

\[ \mathbb{P} \left( \text{Leb} \left( \left\{ t \geq 0 : Z(t) \in \partial (\mathbb{R}^d_+) \right\} \right) = 0 \right) = 1. \]

Another boundary property: for every \( J \subseteq \{1, \ldots, d\} \) s.t. \(|J| \geq 2\), and for any \( k \in J \), we have

\[ \int_0^\infty 1_{\{Z_J(s)=0\}} \, dY_k(s) = 0. \]

Stationary distribution: If the skew-symmetry condition is satisfied, and \( Z \) has a stationary distribution, then it is given by a product of exponentials:

\[ \rho(z) = C(\mu) \exp \{ \gamma(\mu) \cdot z \}. \]
Outline

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Our results
Sticky Brownian motion on the half-line

Governed by the SDE:

$$dX(t) = \lambda 1_{\{X(t)=0\}} \, dt + 1_{\{X(t)>0\}} \, dB(t)$$

- Weak existence and uniqueness hold, but strong existence and pathwise uniqueness fail.
- Arises through a stochastic time change of reflected Brownian motion
- If $B$ has negative drift, $X$ has an exponential stationary distribution with an added mass at zero.

Well studied:

- Feller (1950’s), Itô and McKean (1963)
- Harrison and Lemoine (1981)
- Chitasvili (1989)
- Amir (1991)
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Our results
When apart, particles jump as usual:

When there is a collision, the whole system slows down:
Sticky particles

Let $M > 0$ denote the scaling parameter.

Consider independent Poisson processes $P_i$, $Q_i$, $L_{i,j}$, and $R_{i,j}$:

- all have jump size $1/\sqrt{M}$;
- $P_i$, $Q_i$ have jump rates $Ma$;
- $L_{i,j}$ has jump rate $\sqrt{M}\theta_{i,j}$, $R_{i,j}$ has jump rate $\sqrt{M}\theta_{i,j}$.

The particle system $X^M (\cdot) = (X^M_1 (\cdot), \ldots, X^M_n (\cdot))$ evolves on $\mathbb{Z}/\sqrt{M}$:

$$dX^M_i (t) = \mathbf{1} \left\{ x^M_k (t) + \frac{1}{\sqrt{M}} < x^M_{k+1} (t), k \in [n-1] \right\} d(P_i (t) - Q_i (t))$$

$$+ \sum_{j=1}^{n-1} \mathbf{1} \left\{ x^M_i (t) + \frac{1}{\sqrt{M}} < x^M_{i+1} (t), x^M_j (t) + \frac{1}{\sqrt{M}} = x^M_{j+1} (t) \right\} dR_{i,j} (t)$$

$$- \sum_{j=1}^{n-1} \mathbf{1} \left\{ x^M_{i-1} (t) + \frac{1}{\sqrt{M}} < x^M_i (t), x^M_j (t) + \frac{1}{\sqrt{M}} = x^M_{j+1} (t) \right\} dL_{i,j} (t),$$

and remains ordered:

$$X^M (t) \in \mathcal{W}^M := \left\{ x \in \left( \mathbb{Z}/\sqrt{M} \right)^n : x_k + 1/\sqrt{M} \leq x_{k+1}, k \in [n-1] \right\}.$$
Sticky particles

Important quantities:

- **The speed change matrix** $V = (v_{i,j})$: 
  $$v_{i,j} := \begin{cases} 
  \theta_{i,j}^R - \theta_{i,j}^L & \text{if } j \neq i - 1, i, \\
  \theta_{i,i-1}^R & \text{if } j = i - 1, \\
  -\theta_{i,i}^L & \text{if } j = i, 
  \end{cases}$$

  $v_{i,j}$ = velocity of particle $i$ when particles $j$ and $j + 1$ are adjacent (and no others)

- **The reflection matrix** $Q = (q_{j,j'})$: 
  $$q_{j,j'} = v_{j+1,j} - v_{j,j'},$$

  $q_{j,j'}$ = velocity of gap $j$ when particles $j'$ and $j' + 1$ are adjacent (and no others)
Theorem (R., Shkolnikov (2013))

Suppose that
- $Q$ is completely-$S$, [...], and
- the initial conditions $\{(X_1^M(0), \ldots, X_n^M(0)) \, : \, M > 0\}$ are deterministic and converge to a limit $(x_1, \ldots, x_n) \in \mathcal{W}$ as $M \to \infty$.

Then the laws of the paths of the particle systems $\{(X_1^M(\cdot), X_2^M(\cdot), \ldots, X_n^M(\cdot)) \, : \, M > 0\}$ on $D([0, \infty), \mathbb{R}^n)$ converge to the law of the unique weak solution of the system of SDEs

$$dX_i(t) = 1\{x_1(t)<x_2(t)<\ldots<x_n(t)\} \sqrt{2a} \, dW_i(t) + \sum_{j=1}^{n-1} 1\{x_j(t)=x_{j+1}(t)\} \, v_{i,j} \, dt,$$

$i \in [n]$, in $\mathcal{W}$ starting from $(x_1, \ldots, x_n)$.

Here $W = (W_1, \ldots, W_n)$ is a standard Brownian motion in $\mathbb{R}^n$. 
Assumptions

**Intuitively:** completely-$S$ condition necessary to stay in the wedge $\mathcal{W}$.

Define the $(n - 1) \times (n - 1)^2$ matrix $Q^{(2)} = \left( q_{i,(k,\ell)}^{(2)} \right)$:

$$
q_{i,(k,\ell)}^{(2)} := \begin{cases} 
-\theta_{i,\ell}^L & \text{if } k = i - 1, \ell \neq i - 1, \\
\theta_{i+1,\ell}^L + \theta_{i,\ell}^R & \text{if } k = i, \ell \neq i, \\
-\theta_{i+1,\ell}^R & \text{if } k = i + 1, \ell \neq i + 1, \\
0 & \text{otherwise.}
\end{cases}
$$

**Our additional assumption**

Assume that the matrices $Q$ and $Q^{(2)}$ (restricted to nonzero columns) are "jointly completely-$S$", in the following sense.

For every $J \subseteq [n - 1]$, $J \neq \emptyset$, there exists $\gamma \in (\mathbb{R}_+)^{|J|}$ s.t.

- $\gamma \cdot q_{j,j}^{(2)} \geq 1$ for every $j \in J$, and
- $\gamma \cdot q_{i,(k,\ell)}^{(2)} \geq 1$ for every $k, \ell \in J$ such that $q_{i,(k,\ell)}^{(2)}$ is nonzero.

Satisfied in many natural cases, e.g., when $\theta_{i,j}^L = \theta_{i,j}^R = \theta > 0$. 
Proof steps

In order to prove our convergence result:

- weak existence and uniqueness of the SDE,
- tightness of \( \{ X^M(\cdot), M > 0 \} \),
- identification of the limit.
Consider the system of SDEs

\[ dX_i(t) = 1_{\{x_1(t)<x_2(t)<...<x_n(t)\}} (b_i \, dt + dW_i(t)) + \sum_{j=1}^{n-1} 1_{\{x_j(t)=x_{j+1}(t)\}} v_{i,j} \, dt. \]

Here:
- \( b_i, i \in [n] \) are real constants;
- \( W = (W_1, \ldots, W_n) \) is an \( n \)-dim. Brownian motion w/ zero drift, and a strictly positive definite diffusion matrix \( C = (c_{i,j'}) \);
- \( V = (v_{i,j}) \) is a matrix with real entries;
- the initial conditions \( X_i(0) = x_i, i \in [n] \), satisfy \( (x_1, \ldots, x_n) \in \mathcal{W} \).

**Theorem**

If \( Q \) is completely-\( S \), then there exists a unique weak solution to the system of SDEs above. Moreover, if \( Q \) is not completely-\( S \), then there is no weak solution.

Note: the diffusion matrix of \( X \) is both discontinuous, and degenerate, so classical results do not apply.
Weak existence of the SDE

\[ dX_i(t) = \mathbf{1}_{\{X_1(t) < X_2(t) < \ldots < X_n(t)\}} (b_i \, dt + dW_i(t)) + \sum_{j=1}^{n-1} \mathbf{1}_{\{X_i(t) = X_{i+1}(t)\}} v_{i,j} \, dt. \]

Main ideas for weak existence:
- Start from an appropriate SRBM in the orthant \((\mathbb{R}_+)^{n-1}\), and
- apply an appropriate stochastic time change.
Weak existence of the SDE

\[ dX_i(t) = 1 \{ x_1(t) < x_2(t) < \ldots < x_n(t) \} \left( b_i \, dt + dW_i(t) \right) + \sum_{j=1}^{n-1} 1 \{ x_j(t) = x_{j+1}(t) \} \, v_{i,j} \, dt. \]

Main ideas for weak existence:
- Start from an appropriate SRBM in the orthant \((\mathbb{R}_+)^{n-1}\), and
- apply an appropriate stochastic time change.

Since \( Q \) is completely-\( S \), there exists a weak solution to

\[ d\hat{Z}_j(t) = (b_{j+1} - b_j) \, dt + dB_j(t) + \sum_{j' = 1}^{n-1} q_{j,j'} \, d\Lambda_{j'}(t), \quad j \in [n - 1], \]

- with initial conditions \( \hat{Z}_j(0) = x_{j+1} - x_j \), \( j \in [n - 1] \),
- \( B = (B_1, \ldots, B_{n-1}) \) is a Brownian motion with zero drift and diffusion matrix \( A = (a_{j,j'}) \) given by

\[ a_{j,j'} = c_{j,j'} + c_{j+1,j'+1} - c_{j,j'+1} - c_{j+1,j'}, \]

- \( \Lambda_j(\cdot) \) is the semimartingale local time at zero of \( \hat{Z}_j(\cdot) \), and
- recall that \( q_{j,j'} = v_{j+1,j'} - v_{j,j'} \).
Weak existence of the SDE

- Let \( \hat{\beta} = (\hat{\beta}_1, \hat{\beta}_2, \ldots, \hat{\beta}_n) \) be a Brownian motion with zero drift and diffusion matrix \( \mathbf{C} \) such that \( B_i(\cdot) = \hat{\beta}_{i+1}(\cdot) - \hat{\beta}_i(\cdot) \).

- Define \( \hat{X} = (\hat{X}_1, \hat{X}_2, \ldots, \hat{X}_n) \) as the unique process satisfying
  \[
  \sum_{i=1}^{n} \hat{X}_i(t) = \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \left( b_i t + \hat{\beta}_i(t) + \sum_{j=1}^{n-1} \nu_{i,j} \Lambda_j(t) \right),
  \]
  \[
  (\hat{X}_2(t) - \hat{X}_1(t), \ldots, \hat{X}_n(t) - \hat{X}_{n-1}(t)) = (\hat{Z}_1(t), \ldots, \hat{Z}_{n-1}(t)).
  \]

- Finally, define
  \[
  T(t) := t + \Lambda(t) := t + \sum_{j=1}^{n-1} \Lambda_j(t), \quad t \geq 0,
  \]
  \[
  \tau(t) := \inf\{s \geq 0 : T(s) = t\}, \quad t \geq 0,
  \]
  and apply the stochastic time change
  \[
  X(\cdot) = \hat{X}(\tau(\cdot)).
  \]
Properties of sticky Brownian motion

Behavior at the boundary: does not spend a non-empty time interval on $\partial W$, however,

$$\mathbb{P} \left( \text{Leb} \left( \{ t \geq 0 : X(t) \in \partial W \} \right) > 0 \right) = 1.$$ 

Stationary distributions: under certain conditions, the gap process $Z(\cdot)$ has a stationary distribution as a weighted sum of exponentials.

Suppose $Q^{-1} (b_2 - b_1, b_3 - b_2, \ldots, b_n - b_{n-1})^T < 0$ componentwise and $2A = QD + DQ^T$, where $D = \text{diag}(A)$. Let

$$\gamma = 2D^{-1}Q^{-1} (b_2 - b_1, b_3 - b_2, \ldots, b_n - b_{n-1})^T.$$

Then the stationary distribution of $Z$ in the orthant $(\mathbb{R}_+)^{n-1}$ is

$$\frac{1}{C} e^{\langle \gamma, z \rangle} \left( dz + \sum_{j=1}^{n-1} \frac{\sqrt{a_{j,j}}}{2} 1_{\{z \in F_j\}} dz^j \right).$$
To obtain our convergence result, we study the decomposition

\[ X_i^M(t) = X_i^M(0) + A_i^M(t) + \sum_{j=1}^{n-1} C_{i,j}^{R,M}(t) - \sum_{j=1}^{n-1} C_{i,j}^{L,M}(t) + \sum_{j=1}^{n-1} \Delta_{i,j}^{R,M}(t) - \sum_{j=1}^{n-1} \Delta_{i,j}^{L,M}(t), \]

where

\[ A_i^M(t) := \int_0^t 1 \left\{ X_k^M(s) + \frac{1}{\sqrt{M}} < X_{k+1}^M(s), k \in [n-1] \right\} d(P_i(s) - Q_i(s)), \]

\[ C_{i,j}^{R,M}(t) := \theta_{i,j} I_{i,j}^{R,M}(t) := \theta_{i,j} \int_0^t 1 \left\{ X_i^M(s) + \frac{1}{\sqrt{M}} < X_{i+1}^M(s), X_j^M(s) + \frac{1}{\sqrt{M}} = X_{j+1}^M(s) \right\} ds, \]

\[ \Delta_{i,j}^{R,M}(t) := \int_0^t 1 \left\{ X_i^M(s) + \frac{1}{\sqrt{M}} < X_{i+1}^M(s), X_j^M(s) + \frac{1}{\sqrt{M}} = X_{j+1}^M(s) \right\} d\left( R_{i,j}(s) - \theta_{i,j} s \right). \]

Main technical issue: dealing with the indicator functions.
Convergence statement

Under the assumptions stated before, the family

\[
\left\{ \left( X^M, A^M, I^{L,M}, I^{R,M}, \Delta^{L,M}, \Delta^{R,M} \right) , M > 0 \right\}
\]

is tight in \( D \left[ 0, \infty \right), \mathbb{R}^{4n^2-2n} \). Moreover, every limit point satisfies

\[
X_i^\infty (\cdot) = \int_0^\cdot 1\{ X_1^\infty (s) < \ldots < X_n^\infty (s) \} \sqrt{2a} dW_i (s) + \sum_{j=1}^{n-1} v_{i,j} \int_0^\cdot 1\{ X_j^\infty (s) = X_{j+1}^\infty (s) \} ds,
\]

\[
A_i^\infty (\cdot) = \int_0^\cdot 1\{ X_1^\infty (s) < X_2^\infty (s) < \ldots < X_n^\infty (s) \} \sqrt{2a} dW_i (s),
\]

\[
I_{i,j}^{L,\infty} (\cdot) = \int_0^\cdot 1\{ X_j^\infty (s) = X_{j+1}^\infty (s) \} ds, \quad j \in [n-1] \setminus \{i-1\},
\]

\[
I_{i,j}^{R,\infty} (\cdot) = \int_0^\cdot 1\{ X_j^\infty (s) = X_{j+1}^\infty (s) \} ds, \quad j \in [n-1] \setminus \{i\},
\]

\[
I_{i,i-1}^{L,\infty} (\cdot) = I_{i,i}^{R,\infty} (\cdot) = 0,
\]

\[
\Delta_{i,j}^{L,\infty} (\cdot) = \Delta_{i,j}^{R,\infty} (\cdot) = 0, \quad j \in [n-1],
\]

with a suitable \( n \)-dimensional standard Brownian motion \( W = (W_1, \ldots, W_n) \).
Step 1. Tightness

Step 2. A few observations:

- The jumps of all components are bounded by $1/\sqrt{M}$, so all components of the limit point must have continuous paths.
- The family $\{A^M(t), M > 0\}$ is uniformly integrable:

$$
\mathbb{E} \left[ A_i^M(t)^2 \right] = \mathbb{E} \left[ \int_0^t 1 \left\{ X_k^M(s) + \frac{1}{\sqrt{M}} X_{k+1}^M(s), k \in [n-1] \right\} \frac{1}{\sqrt{M}} (dP_i + dQ_i)(s) \right] \leq 2at,
$$

and since $A^M$ is a martingale for any fixed $M > 0$, $A^\infty$ is a martingale w.r.t. its own filtration.
- As limits of non-decreasing processes, $I_{i,j}^{L,\infty}$ and $I_{i,j}^{R,\infty}$ are non-decreasing themselves, and so are of finite variation.
- For every $t \geq 0$ we have

$$
\lim_{M \to \infty} \mathbb{E} \left[ \left[ \Delta_{i,j}^{L,M} \right](t) \right] = \lim_{M \to \infty} \mathbb{E} \left[ \left[ \Delta_{i,j}^{R,M} \right](t) \right] = 0,
$$

so $\Delta_{i,j}^{L,\infty} \equiv \Delta_{i,j}^{R,\infty} \equiv 0$.
- The quadratic covariation processes $\langle X_i^\infty, X_i^\infty \rangle$ and $\langle A_i^\infty, A_i^\infty \rangle$ are equal.
Step 3. Determining the quadratic covariation processes

\[ \langle X_i^\infty, X_i'^\infty \rangle = \langle A_i^\infty, A_i'^\infty \rangle. \]
Convergence proof sketch, II

Step 3. Determining the quadratic covariation processes

\[ \langle X_i^\infty, X_j^\infty \rangle = \langle A_i^\infty, A_j^\infty \rangle. \]

First: \( \langle A_i^\infty, A_j^\infty \rangle = 0 \) when \( i \neq j' \).

- For any \( t \geq 0 \), the family \( \{ A_i^M(t)A_j^{M'}(t), M > 0 \} \) is uniformly integrable due to \( \mathbb{E}[A_i^M(t)^2A_j^{M'}(t)^2] \leq 4a^2t^2 \);  
- \( A_i^\infty(\cdot)A_j^\infty(\cdot) \) is the limit in \( D([0, \infty), \mathbb{R}) \) of the family of martingales \( \{ A_i^M(\cdot)A_j^{M'}(\cdot), M > 0 \} \);  
- So \( A_i^\infty(\cdot)A_j^\infty(\cdot) \) is a martingale w.r.t. its own filtration.
Convergence proof sketch, II

Step 3. Determining the quadratic covariation processes

\[ \langle X_i^\infty, X_{i'}^\infty \rangle = \langle A_i^\infty, A_{i'}^\infty \rangle. \]

First: \( \langle A_i^\infty, A_{i'}^\infty \rangle = 0 \) when \( i \neq i' \).

- For any \( t \geq 0 \), the family \( \{ A_i^M(t) A_{i'}^M(t), M > 0 \} \) is uniformly integrable due to \( \mathbb{E} [A_i^M(t)^2 A_{i'}^M(t)^2] \leq 4a^2 t^2; \)
- \( A_i^\infty(\cdot)A_{i'}^\infty(\cdot) \) is the limit in \( D([0, \infty), \mathbb{R}) \) of the family of martingales \( \{ A_i^M(\cdot)A_{i'}^M(\cdot), M > 0 \} \);
- So \( A_i^\infty(\cdot)A_{i'}^\infty(\cdot) \) is a martingale w.r.t. its own filtration.

Second: \( \langle X_i^\infty \rangle(\cdot) = \langle A_i^\infty \rangle(\cdot) = 2a \int_0^\cdot 1_{\{X_i^\infty(s) < \ldots < X_n^\infty(s)\}} \, ds \)

- The family \( \{ A_i^M(t)^2, M > 0 \} \) is uniformly integrable;
- Use Portmanteau with the lower semicontinuity of
  \[ (\omega_1, \omega_2, \ldots, \omega_n) \mapsto \int_{t_1}^{t_2} 1_{\{\omega_1(s) < \ldots < \omega_n(s)\}} \, ds; \]
- Use the occupation time formula to show that the measure \( d \langle X_i^\infty \rangle \) assigns zero mass to the sets \( \{ t \geq 0 : X_j^\infty(t) = X_j^{\infty}(t) \}. \)
Convergence proof sketch, II

Step 4. Dealing with the finite variation terms

Want to show that

\[ I_{i,j}^{\infty,2}(\cdot) := \int_0^\cdot 1 \{ X_{i+1}(s) = X_i(s), X_{j+1}(s) = X_j(s) \} \, ds \]

is identically zero when \( i \neq j \).

The limiting processes inherit many properties of the prelimit processes, and after the stochastic time change the process of spacings can be written as

\[ \hat{Z}(\cdot) = \hat{Z}(0) + \hat{B}(\cdot) + Q I_{\infty,1}^{\infty,1}(\cdot) + Q^{(2)} I_{\infty,2}^{\infty,2}(\cdot). \]

Can finish the argument as Reiman and Williams (1988) did for the boundary property of SRBM, but need to assume that \( Q \) and \( Q^{(2)} \) are “jointly completely-S”.
More general particle systems

\[
\begin{align*}
\frac{dX_i^M(t)}{dt} &= \frac{1}{\sqrt{M}} \mathbf{1}\{X_k^M(t)+\frac{1}{\sqrt{M}} < X_{k+1}^M(t), k \in [n-1]\} \, d\left(S_i^R(M \, t) - S_i^L(M \, t)\right) \\
&\quad + \frac{1}{\sqrt{M}} \sum_{j=1}^{n-1} \mathbf{1}\{X_i^M(t)+\frac{1}{\sqrt{M}} < X_{i+1}^M(t), X_j^M(t)+\frac{1}{\sqrt{M}} = X_{j+1}^M(t)\} \, dT_{i,j}^R(\sqrt{M} \, t) \\
&\quad - \frac{1}{\sqrt{M}} \sum_{j=1}^{n-1} \mathbf{1}\{X_{i-1}^M(t)+\frac{1}{\sqrt{M}} < X_i^M(t), X_j^M(t)+\frac{1}{\sqrt{M}} = X_{j+1}^M(t)\} \, dT_{i,j}^L(\sqrt{M} \, t),
\end{align*}
\]

- $S_i^L$ and $S_i^R$, $i \in [n]$, are renewal processes, not necessarily independent,
- $T_{i,j}^L$ and $T_{i,j}^R$, $i \in [n]$, $j \in [n-1]$, are independent renewal processes.
More general particle systems

Under the previous assumptions and an additional moment assumption on the interarrival times between jumps, we have: The laws of the paths of the particle systems \( \{X^M(\cdot), M > 0\} \) converge in \( D([0, \infty), \mathbb{R}^n) \) to the law of the unique weak solution of the system of SDEs

\[
\begin{align*}
    dX_i(t) &= 1\{x_1(t)<\ldots<x_n(t)\} \left( (\lambda_i^R - \lambda_i^L) \, dt + a^{3/2} \, dW_i(t) \right) \\
    &\quad + \sum_{j=1}^{n-1} 1\{x_j(t)=x_{j+1}(t)\} \, v_{i,j} \, dt
\end{align*}
\]

for \( i \in [n] \), taking values in \( \mathcal{W} \) and starting from \( x \). Here, \( \mathcal{W} = (W_1, \ldots, W_n) \) is a Brownian motion in \( \mathbb{R}^n \) with zero drift and diffusion matrix given by

\[
\mathcal{C} = (c_{i,j}) = (c_{i,i}^L + c_{i,i}^R + c_{i,i}^{L,R} + c_{i,i}^{R,R}),
\]

where these terms come from the covariances of the random variables describing the interarrival times related to the renewal processes \( S_i^L \) and \( S_i^R \), \( i \in [n] \).
Summary and open questions

Takeaways:

- **Exclusion processes with stickiness** (global slowdown)

\[ \downarrow \]

Diffusive limit

Multidimensional sticky Brownian motion

- **Potential applications**
  - Stochastic portfolio theory
  - Queueing theory
Takeaways:

- **Exclusion processes with stickiness** (global slowdown)

  \[ \downarrow \text{diffusive limit} \]

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Open questions:

- Other types of sticky interaction?
- In particular, *local slowdown*?
Summary and open questions

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Thank you!