Finding cliques in random graphs by adaptive probing

Based on joint works with U. Feige, D. Gamarnik, J. Neeman, B. Schiffer, and P. Tetali

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Finding cliques

Erdős-Rényi random graph $G(n, 1/2)$
Finding cliques

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Largest clique $\omega(G) \approx 2 \log n$
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Greedy algorithm: finds clique of size $\log n$
Karp (1976): $\geq (1 + \varepsilon) \log n$
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Planted clique model $G(n, 1/2, k)$

Planted clique of size $k$

Challenge: find planted clique efficiently
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Planted clique model $G(n, 1/2, k)$

- Planted clique of size $k$
- Challenge: find planted clique efficiently
- Information-theoretically possible when $k \geq (2 + \varepsilon) \log n$
- Efficient algorithm known only when $k = \Omega(n^{1/2})$
- Conjectured information-computation gap
Adaptive probing

Goal: find max clique
Constraint: computational efficiency
Adaptive probing

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Constraint: can only look at a small part of the graph
Adaptive probing

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Probe model:
adaptively query pairs of vertices, learn if they are connected by an edge or not

1. \((i_1, j_1) \in E?\)
2. \((i_2, j_2) \in E?\)
3. \((i_3, j_3) \in E?\)
4. Etc.
At most \(q\) queries in total.
Finding cliques by adaptive probing

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Related work on finding structure in a random graph using adaptive edge queries

- Ferber, Krivelevich, Sudakov, Vieira (RSA 2016): Hamilton cycles
- Ferber, Krivelevich, Sudakov, Vieira (RSA 2017): long paths
- Conlon, Fox, Grinshpun, He (2018): target graph $H$ (e.g., small clique)

Main difference: dense vs. sparse random graphs
Erdős-Rényi random graph $G(n, 1/2)$

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Parametrize $q = n^\delta$, wlog $1 \leq \delta < 2$
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Parametrize $q = n^{\delta}$, wlog $1 \leq \delta < 2$

$$\alpha_*(\delta) := \sup \alpha \text{ s.t. there exists algorithm making } \leq n^\delta \text{ adaptive queries that finds a clique of size at least } \alpha \log n \text{ (w/prob. } \geq 1/2)$$
Finding cliques by adaptive probing

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\[ \alpha_*(\delta) \leq 2 \]
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Algorithms?

1. Greedy: finds clique of size $\log n$
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Algorithms?
1. **Greedy**: finds clique of size $\log n$
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3. Combined: greedy until $\sqrt{q}$ nodes remain,
then switch to exhaustive search:

$$\log(n/\sqrt{q}) + 2 \log \sqrt{q} = \log n + \frac{1}{2} \log q = (1 + \delta/2) \log n$$
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$$\alpha_\star(\delta) := \sup \alpha \text{ s.t.}
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$$\log(n/\sqrt{q}) + 2 \log \sqrt{q} = \log n + \frac{1}{2} \log q = (1 + \delta/2) \log n$$

Open problem: find $\alpha_\star(\delta)$

$$\frac{\delta}{2} \leq \alpha_\star(\delta) \leq 2$$
Adaptive probing w/ few rounds

Erdős-Rényi random graph $G(n, 1/2)$

Largest clique

$\omega(G) \approx 2 \log n$

Challenge:
find max clique using
$\leq q$ adaptive edge queries
in at most $\ell$ rounds

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Adaptive probing w/ few rounds

**Erdős-Rényi random graph** $G(n, 1/2)$

- **Largest clique** $\omega(G) \approx 2 \log n$
- **Challenge:** find max clique using $\leq q$ adaptive edge queries in at most $\ell$ rounds

Parametrize $q = n^\delta$, wlog $1 \leq \delta < 2$

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$\alpha_*(\delta, \ell) \leq \alpha_*(\delta) \leq 2$

**Theorem (Feige, Gamarnik, Neeman, R., Tetali 2018)**

For every $\delta < 2$ and constant $\ell$ we have that $\alpha_*(\delta, \ell) < 2$. 
Specific bounds ($\delta = 1$)

Erdős-Rényi random graph $G(n, 1/2)$

Largest clique $\omega(G) \approx 2 \log n$

Challenge: find max clique using $O(n)$ adaptive edge queries in at most $\ell$ rounds

Theorem (Feige, Gamarnik, Neeman, R., Tetali 2018)

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<tr>
<th>Rounds</th>
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<td>$4/3 \leq \alpha_*(1,2) \leq 2^{2/3} &lt; 1.588$</td>
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- Largest clique $\omega(G) \approx 2 \log n$
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Q: Is 3 rounds more powerful than 2 rounds?
Erdős-Rényi random graph $G(n, 1/2)$

1 round: pick $\sqrt{n}$ vertices, probe all pairs
Erdős-Rényi random graph $G(n, 1/2)$

1 round: pick $\sqrt{n}$ vertices, probe all pairs

2 rounds:
Round #1:
- Probe all pairs within $S$.
- Probe all pairs between $S$ and $T$.

Challenge:
find max clique using $O(n)$ adaptive edge queries in at most $\ell$ rounds

$|S| = n^{1/6}$

$|T| = n^{5/6}$
**Challenge:** find max clique using $O(n)$ adaptive edge queries in at most $\ell$ rounds

Erdős-Rényi random graph $G(n, 1/2)$

1 **round:** pick $\sqrt{n}$ vertices, probe all pairs

2 **rounds:**

   **Round #1:**
   - Probe all pairs within $S$.
   - Probe all pairs between $S$ and $T$.
   - $S' :=$ largest clique in $S$, $|S'| \approx \frac{1}{3} \log n$.

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$S'$

$T$
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- $S' :=$ largest clique in $S$, $|S'| \approx \frac{1}{3} \log n$.
- $T' :=$ vertices in $T$ which are connected to every vertex in $S'$.
- $|T'| \approx \frac{n^{5/6}}{2^{(1/3) \log n}} = n^{1/2}$.

Challenge:
find max clique using $O(n)$ adaptive edge queries in at most $\ell$ rounds

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Erdős-Rényi random graph $G(n, 1/2)$

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**Round #2:**
- Probe all pairs within $T'$. 

$|S| = n^{1/6} \quad |T| = n^{5/6}$
Erdős-Rényi random graph $G(n, 1/2)$

**1 round:** pick $\sqrt{n}$ vertices, probe all pairs

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**Round #2:**
- Probe all pairs within $T'$.
- Find clique of size $\approx 2 \log \sqrt{n} = \log n$. 

Challenge: find max clique using $O(n)$ adaptive edge queries in at most $\ell$ rounds
Erdős-Rényi random graph \( G(n, 1/2) \)

**Challenge:** find max clique using \( O(n) \) adaptive edge queries in at most \( \ell \) rounds

**1 round:** pick \( \sqrt{n} \) vertices, probe all pairs

**2 rounds:**

*Round #1:*
- Probe all pairs within \( S \).
- Probe all pairs between \( S \) and \( T \).
- \( S' := \) largest clique in \( S \), \( |S'| \approx \frac{1}{3} \log n \).
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- \( |T'| \approx \frac{n^{5/6}}{2^{(1/3) \log n}} = n^{1/2} \).

*Round #2:*
- Probe all pairs within \( T' \).
- Find clique of size \( \approx 2 \log \sqrt{n} = \log n \).
- Altogether: clique of size \( \frac{4}{3} \log n \).
Erdős-Rényi random graph $G(n, 1/2)$

**Challenge:** find max clique using $O(n)$ adaptive edge queries in at most $\ell$ rounds

**1 round:** pick $\sqrt{n}$ vertices, probe all pairs

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  - $T' :=$ vertices in $T$ which are connected to every vertex in $S'$.
  - $|T'| \approx \frac{n^{5/6}}{2(1/3) \log n} = n^{1/2}$.

- **Round #2:**
  - Probe all pairs within $T'$.
  - Find clique of size $\approx 2 \log \sqrt{n} = \log n$.
  - Altogether: clique of size $\frac{4}{3} \log n$.

**3 rounds:** similar. Exercise!
Ideas about $\alpha_*(\delta, \ell) < 2$

Theorem (Feige, Gamarnik, Neeman, R., Tetali 2018)

For every $\delta < 2$ and constant $\ell$ we have that $\alpha_*(\delta, \ell) < 2$.

**In short:** first moment method + some extremal graph theory
Ideas about $\alpha_\star(\delta, \ell) < 2$

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**In short:** first moment method + some extremal graph theory

**In more detail:**
- Algorithm takes $\ell$ rounds, $O(n)$ queries in each round
- $k := \alpha \log n$; $K$ a set of vertices of size $k$
- Fix $\beta_1, \ldots, \beta_\ell \geq 0$ s.t. $\sum_{i=1}^\ell \beta_i = 1$; will optimize over later

**Def:** Round $i$ is significant if the # of probes to $K$
- in rounds 1 to $i - 1$ is $\leq \sum_{j=1}^{i-1} \beta_j \binom{k}{2}$, and
- in rounds 1 to $i$ is $\geq \sum_{j=1}^i \beta_j \binom{k}{2}$.

**Claim:** there is a significant round.
(Proof: induction.)

- Such a $K$ called an $i$-eligible set.
- Can determine after round $i - 1$.
- After round $i - 1$,

$$P(K \text{ is a clique}) \leq 2^{-\sum_{j=1}^\ell \beta_j \binom{k}{2}}$$

- Union bound over all such $K$.
- To bound their number: extremal graph theory, next slide
An extremal problem

**Def:** $N_{n,m,k,\beta} := \max \# \text{ sets of size } k \text{ that can be } \beta\text{-covered in an } n \text{ vertex graph with } m \text{ edges}$

**Theorem (Feige, Gamarnik, Neeman, R., Tetali 2018)**

When $\beta \in \left[0, \frac{16}{25}\right]$:  

$$N_{n,m,k,\beta} \leq m^{(1-\sqrt{1-\beta})k+1} \cdot n^{(2\sqrt{1-\beta}-1)k+2}.$$  

When $\beta \in \left[\frac{16}{25}, 1\right]$:  

$$N_{n,m,k,\beta} \leq m^{(\sqrt{\beta}/2)k+1} \cdot n^{(1-\sqrt{\beta})k+2}.$$  

**Extremal graphs:**

- Clique + isolated vertices  
- "Complete split graph"
Finding a planted clique via probing

Planted clique model $G(n, 1/2, k)$

Planted clique of size $k$

Challenge:
find planted clique using
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Finding a planted clique via probing

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$$q = n^\delta$$
$$k = n^{\gamma}$$

(R., Schiffer, 2019)
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If $q = o(n^2/k^2)$ then whp no queries contain two planted clique nodes

$\begin{align*}
q &= n^\delta \\
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\end{align*}$

(R., Schiffer, 2019)
Finding a planted clique via probing

Planted clique model $G(n, 1/2, k)$

Challenge: find planted clique using $\leq q$ adaptive edge queries

Algorithm:
1. Sample, find large clique
2. Extend to planted clique

If $q = o(n^2/k^2)$ then whp no queries contain two planted clique nodes

Detection impossible

Detection possible

Recovery possible

Recovery impossible

(R., Schiffer, 2019)
• Adaptive edge query model: constraint worth exploring

• **Main result:** cannot find the largest clique w/ constant rounds

• **Open problem:** compute $\alpha_*(\delta)$. Is $\alpha_*(\delta) < 2$?

• Is three rounds more powerful than two rounds?
Summary

• Adaptive edge query model: constraint worth exploring

• **Main result:** cannot find the largest clique w/ constant rounds

• **Open problem:** compute $\alpha_\star(\delta)$. Is $\alpha_\star(\delta) < 2$?

• Is three rounds more powerful than two rounds?

Thank you!